Frequency domain criteria

Outline

Modeling of a step signal

Unit pulse

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Unit step

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Modeled as pulse response

$$u(k) = \frac{1}{1 - \mathrm{q}^{-1}} \delta(k)$$

Check!

$$\begin{aligned} u(k) &= u(k-1) + \delta(k) \\ u(0) &= u(-1) + \delta(0) = 0 + 1 = 1 \\ u(1) &= u(0) + \delta(1) = 1 + 0 = 1 \\ u(2) &= u(1) + \delta(2) = 1 + 0 = 1 \\ &\vdots \end{aligned}$$

Unstable, since pole $\lambda = 1$

$$\begin{aligned} u &\not\to 0, & k &\to \infty \\ u(k) &= 1, & k &\geq 0 \end{aligned}$$

Asymptotic step response

Step response for system $G(\mathrm{q}^{-1})$

$$\begin{aligned} y(k) &= G(\mathrm{q}^{-1})\tfrac{1}{1-\mathrm{q}^{-1}}\delta(k) = \\ &= G(1)\tfrac{1}{1-\mathrm{q}^{-1}}\delta(k) + \underbrace{[G(\mathrm{q}^{-1}) - G(1)]\frac{1}{1-\mathrm{q}^{-1}}\delta(k)}_{\to 0} \end{aligned}$$

Second term stable!

$$y(k) \to G(1), \quad k \to \infty$$

Example

$$G(q^{-1}) = \frac{2q^{-1} - q^{-2}}{1 - 0.9q^{-1}}$$

Verify that $1 - q^{-1}$ factor in

$$G(q^{-1}) - G(1) = \ldots = \frac{(1 - q^{-1})(-10 + q^{-1})}{1 - 0.9q^{-1}}$$

$$\begin{aligned} y(k) &= G(q^{-1})\tfrac{1}{1-q^{-1}}\delta(k) = \\ &= G(1)\tfrac{1}{1-q^{-1}}\delta(k) + [G(q^{-1}) - G(1)]\tfrac{1}{1-q^{-1}}\delta(k) \\ &= \underbrace{G(1)\frac{1}{1-q^{-1}}\delta(k)}_{\to G(1)=10} + \underbrace{\frac{-10+q^{-1}}{1-0.9q^{-1}}\delta(k)}_{\to 0} \end{aligned}$$

Modeling of a sinusoid signal

Sinusoid $u(k) = \cos(\omega k + \theta)$
Introduce complex signal

$$u_c(k) = \begin{cases} \mathrm{e}^{\mathrm{i}(\omega k+\theta)} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Then $u(k) = \Re u_c(k)$
Modeled as pulse response

$$u_c(k) = \frac{\mathrm{e}^{\mathrm{i}\theta}}{1 - \mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k)$$

Check!

$$\begin{aligned}
&u_c(k) = \mathrm{e}^{\mathrm{i}\omega} u_c(k-1) + \mathrm{e}^{\mathrm{i}\theta}\delta(k) \\
&u_c(0) = \mathrm{e}^{\mathrm{i}\omega} u_c(-1) + \mathrm{e}^{\mathrm{i}\theta}\delta(0) = \mathrm{e}^{\mathrm{i}\theta} \\
&u_c(1) = \mathrm{e}^{\mathrm{i}\omega} u_c(0) + \mathrm{e}^{\mathrm{i}\theta}\delta(1) = \mathrm{e}^{\mathrm{i}(\omega+\theta)} \\
&u_c(2) = \mathrm{e}^{\mathrm{i}\omega} u_c(1) + \mathrm{e}^{\mathrm{i}\theta}\delta(2) = \mathrm{e}^{\mathrm{i}(\omega 2+\theta)} \\
&\vdots \\
&u_c(k) = \mathrm{e}^{\mathrm{i}(\omega k+\theta)}
\end{aligned}$$

Unstable pole $|\lambda| = |\mathrm{e}^{\mathrm{i}\omega}| = 1$

$$\begin{aligned}
&u_c \not\to 0, \quad k \to \infty \\
&u(k) = \Re u_c(k) = \cos(\omega k + \theta), \quad k \geq 0
\end{aligned}$$

Asymptotic sinusoid response

System response to input $u(k) = \Re u_c(k) = \cos(\omega k + \theta)$
Complex response

$$\begin{aligned} y_c(k) &= G(\mathrm{q}^{-1})u_c(k) = G(\mathrm{q}^{-1})\tfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k) = \\ &= G(\mathrm{e}^{-\mathrm{i}\omega})\tfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k) + \underbrace{[G(\mathrm{q}^{-1}) - G(\mathrm{e}^{-\mathrm{i}\omega})])\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k)}_{\to 0} \end{aligned}$$

Real response $y(k) = \Re y_c(k)$

$$\begin{aligned} & y(k) \to \Re G(\mathrm{e}^{-\mathrm{i}\omega})\tfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k) = |G(\mathrm{e}^{-\mathrm{i}\omega})|\Re\tfrac{\mathrm{e}^{\mathrm{i}(\theta+\arg G(\mathrm{e}^{-\mathrm{i}\omega}))}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\delta(k) \\ & = |G(\mathrm{e}^{-\mathrm{i}\omega})|\cos(\omega k + \theta + \arg G(\mathrm{e}^{-\mathrm{i}\omega})) \end{aligned}$$

Frequency response function

$$G(\mathrm{e}^{-\mathrm{i}\omega}) = |G(\mathrm{e}^{-\mathrm{i}\omega})|\mathrm{e}^{\mathrm{i}\,\arg G(\mathrm{e}^{-\mathrm{i}\omega})}$$

Polar representation can be found experimentally

$$u \longrightarrow \boxed{G} \longrightarrow y$$

$$\cos(\omega k) \qquad\qquad |G(\mathrm{e}^{-\mathrm{i}\omega})| \cos(\omega k + \arg(G(\mathrm{e}^{-\mathrm{i}\omega})))$$

Frequency analysis experiment, for ω_k, $k = 1, 2, \ldots, N$

1. Choose a sinusoidal input of frequency ω_k
2. After transient, measure amplitude and phase shift

Bode curves

Amplitude: Plot $|G(\mathrm{e}^{-\mathrm{i}\omega})|$ against ω

Phase: Plot $\arg G(\mathrm{e}^{-\mathrm{i}\omega})$ against ω

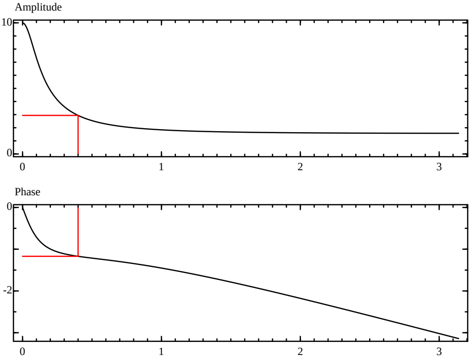
Example: Bode curves

$$G(q^{-1}) = \frac{2q^{-1} - q^{-2}}{1 - 0.9q^{-1}}$$

```
B=[0 2 -1];
A=[1 -0.9];
w=0:0.01:pi;
e=exp(-i*w);
Bw=polyval(fliplr(B),e);
Aw=polyval(fliplr(A),e);
Gw=Bw./Aw;

subplot 211
title('Amplitude')
plot(w,abs(Gw));
g04=abs(Gw(41));
plot([0 0.4 0.4],[g04 g04 0],'r');

subplot 212
title('Phase')
plot(w,angle(Gw));
ang04=angle(Gw(41))
plot([0 0.4 0.4],[ang04 ang04 0],'r');
```

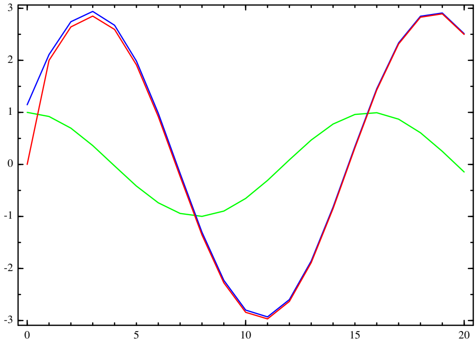


Example: response

Response for frequency $\omega_1 = 0.4$
Input $u(k) = \cos(\omega_1 k)$
Output y and asymptotic output y_∞

$$y(k) \to y_\infty(k) = |G(\mathrm{e}^{-\mathrm{i}\omega_1})| \cos(\omega_1 k + \arg G(\mathrm{e}^{-\mathrm{i}\omega_1}))$$

```
clf;
k=0:20;
u=cos(0.4*k);
yinf=g04*cos(0.4*k+ang04);
y=filter(B,A,u);
plot(k,u,'g');
plot(k,yinf,'b');
plot(k,y,'r');
```



Low pass

$$G(\mathrm{q}^{-1}) = \frac{0.1\mathrm{q}^{-1}}{1 - 0.9\mathrm{q}^{-1}}$$

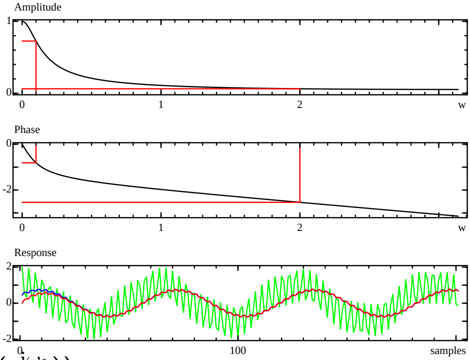
Compare frequencies

$$\omega_1 << \omega_2$$

$$u(k) = \cos(\omega_1 k) + \cos(\omega_2 k)$$

Pass of low frequency

$$y(k) \approx |G(\mathrm{e}^{\mathrm{i}\omega_1})| \cos(\omega_1 k + \arg(G(\mathrm{e}^{\mathrm{i}\omega_1}))$$



High pass

$$G(\mathrm{q}^{-1}) = 0.5(1 - \mathrm{q}^{-1})$$

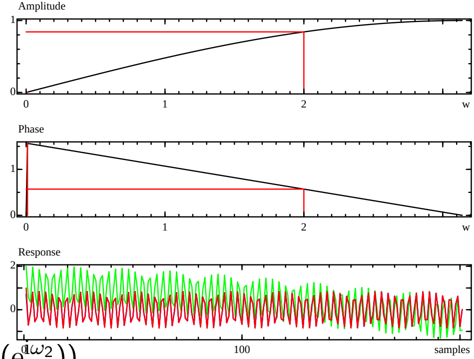
Compare frequencies

$$\omega_1 << \omega_2$$

$$u(k) = \cos(\omega_1 k) + \cos(\omega_2 k)$$

Pass of high frequency

$$y(k) \approx |G(\mathrm{e}^{\mathrm{i}\omega_2})| \cos(\omega_2 k + \arg(G(\mathrm{e}^{\mathrm{i}\omega_2}))$$



Root location and change of argument

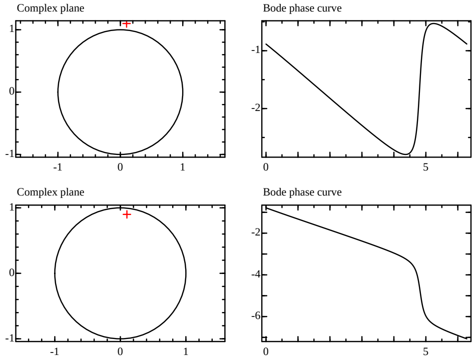
$$A(\mathrm{q}^{-1}) = a_0 + a_1\mathrm{q}^{-1} + \ldots + a_n\mathrm{q}^{-n} = a_n(\mathrm{q}^{-1} - \rho_1)\ldots(\mathrm{q}^{-1} - \rho_n)$$

Roots $\rho = \frac{1}{\lambda}$, $A(\rho) = 0$. Stable pole $|\lambda| < 1 \Rightarrow |\rho| > 1$

Study total change of argument

$$\arg(\mathrm{e}^{-\mathrm{i}\omega} - \rho), \quad \omega = 0 \rightarrow 2\pi$$

$$|\rho| > 1 \Rightarrow \Delta_{\arg}(\mathrm{e}^{-\mathrm{i}\omega} - \rho) = 0$$
$$|\rho| < 1 \Rightarrow \Delta_{\arg}(\mathrm{e}^{-\mathrm{i}\omega} - \rho) = -2\pi$$



Stable polynomial

If all poles $|\lambda_k| < 1$ then

$$\Delta_{\text{arg}} A(\mathrm{e}^{-\mathrm{i}\omega}) = \underbrace{\Delta_{\text{arg}}(\mathrm{e}^{-\mathrm{i}\omega} - \rho_1)}_{=0} + \ldots + \underbrace{\Delta_{\text{arg}}(\mathrm{e}^{-\mathrm{i}\omega} - \rho_n)}_{=0} = 0$$

The modified Mikhaylov criterion

A system with characteristic polynomial $A(\mathrm{q}^{-1})$ is stable if and only if the curve $A(\mathrm{e}^{-\mathrm{i}\omega})$, $\omega = 0 \rightarrow 2\pi$ does not encircle the origin.

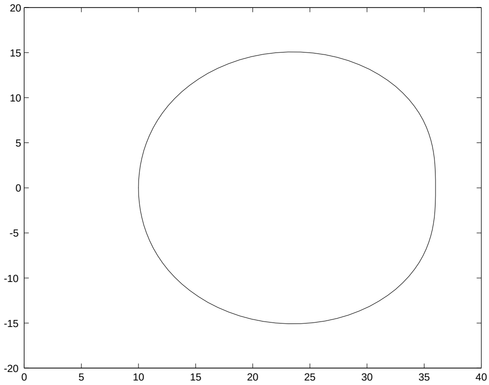
Example

Stable poles?

$$A(\mathrm{q}^{-1}) = 24 - 14\mathrm{q}^{-1} - \mathrm{q}^{-2} + \mathrm{q}^{-3}$$

Mikhaylov locus

$$A(\mathrm{e}^{-\mathrm{i}\omega}), \quad \omega = 0 \to 2\pi$$



Necessary conditions for stability

Necessary condition $(\omega = 0, \pi)$

$$signA(1) = signA(-1)$$

otherwise origin is enclosed

$$A(\mathrm{q}^{-1}) = 1 + \ldots + a_n \mathrm{q}^{-n}$$
$$= (1 - \lambda_1 \mathrm{q}^{-1}) \ldots (1 - \lambda_n \mathrm{q}^{-1})$$
$$\Rightarrow |a_n| = |\lambda_1 \lambda_2 \ldots \lambda_n|$$

Necessary condition

$$|a_n| < 1$$

since $|\lambda_k| < 1, \forall k$

Necessary conditions (easy check!)

$$signA(1) = signA(-1)$$
$$|a_n| < 1$$

Example

$$A(\mathrm{q}^{-1}) = 24 - 14\mathrm{q}^{-1} - \mathrm{q}^{-2} + \mathrm{q}^{-3}$$
$$A(1) = 10$$
$$A(-1) = 36$$
$$A/24 = 1 \ldots \frac{1}{24}\mathrm{q}^{-3} \Rightarrow a_3 = \frac{1}{24}$$

Necessary conditions ok!

Example

$$A(\mathrm{q}^{-1}) = 1 - 1.1\mathrm{q}^{-1} + 1.16\mathrm{q}^{-2} - 0.106\mathrm{q}^{-3}$$

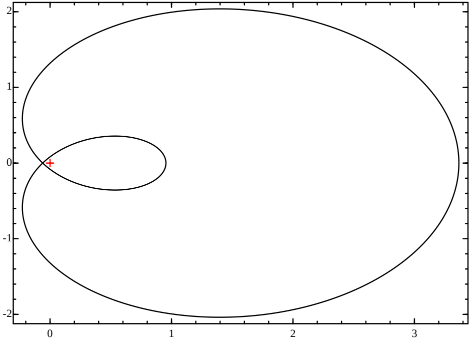
Check:

$$A(1) = 0.954$$
$$A(-1) = 3.366$$
$$a_3 = -0.106$$

Necessary conditions ok!

$$signA(1) = signA(-1), |a_3| < 1$$

But...



Origin encircled twice

In fact: $\lambda_{1,2} = 0.5 \pm 0.9i,\ \lambda_3 = 0.1$

Second order system

$$A(\mathrm{q}^{-1}) = 1 + a_1\mathrm{q}^{-1} + a_2\mathrm{q}^{-2} \quad \begin{aligned} &= (1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1}) \\ &= 1-(\lambda_1+\lambda_2)\mathrm{q}^{-1} + \lambda_1\lambda_2\mathrm{q}^{-2} \end{aligned}$$

Start with $\lambda_1 = \lambda_2 = 0$ (stable)

Move poles to make system unstable

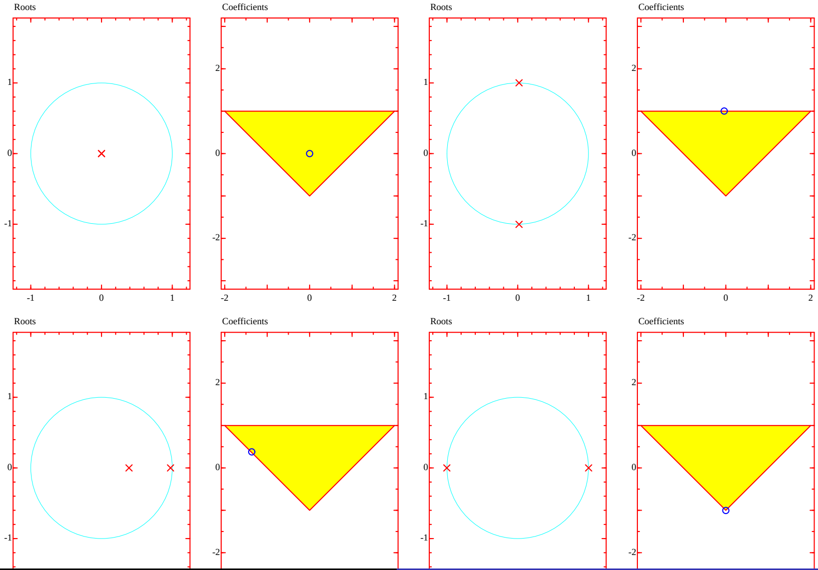
Only three situations possible

1. A pole pass out at $\lambda_1 = 1 \Rightarrow A(1) = 0 = 1 + a_1 + a_2$
2. A pole pass out at $\lambda_1 = -1 \Rightarrow A(-1) = 0 = 1 - a_1 + a_2$
3. Two poles pass out $\lambda_1\lambda_2 = a_2 = 1$

Three lines enclose stability area

$$\begin{cases} a_2 > -a_1 - 1 \\ a_2 > a_1 - 1 \\ a_2 < 1 \end{cases}$$

Example: second order system



Open and closed-loop relation

Process $Ay = Bu$, $y = \frac{B}{A}u$
Controller $Ru = Se$, $u = \frac{S}{R}e$
Open loop compensated system (e external)

$$y = G_{open}e, \quad G_{open} = \frac{BS}{AR} = \frac{B_{open}}{A_{open}}$$

Feedback system $e = -y$

$$ARy = BRu = B(-Sy) \Rightarrow \underbrace{(AR + BS)}_{A_{closed}} y = 0$$

Relation between open and closed loop characteristic polynomials

$$1 + G_{open} = \frac{A_{closed}}{A_{open}}$$

Nyquist theorem

$$A_{closed} = (1 + G_{open})A_{open}$$

Suppose open system has N_u unstable poles
Closed loop stable if

$$\begin{aligned} 0 &= \Delta_{\text{arg}} A_{closed}(\mathrm{e}^{-\mathrm{i}\omega}) \\ &= \Delta_{\text{arg}}(1 + G_{open}(\mathrm{e}^{-\mathrm{i}\omega})) + \underbrace{\Delta_{\text{arg}} A_{open}(\mathrm{e}^{-\mathrm{i}\omega})}_{-2\pi N_u} \end{aligned}$$

$$\Rightarrow \Delta_{\text{arg}}(1 + G_{open}(\mathrm{e}^{-\mathrm{i}\omega})) = 2\pi N_u$$

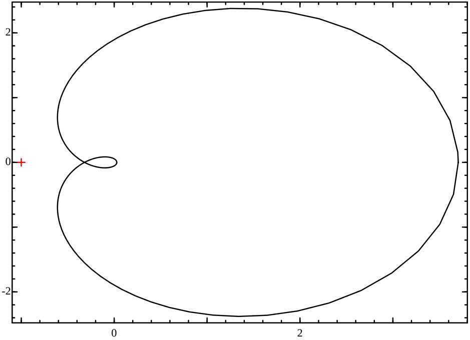
Nyquist theorem

An open-loop system G_{open} with N_u unstable poles is stable in closed loop if and only if $G_{open}(\mathrm{e}^{-\mathrm{i}\omega})$, $\omega = 0 \to 2\pi$ encircles -1, N_u times in counter-clockwise direction.

Example

Stable open loop

$$G_{open}(q^{-1}) = \frac{0.1q^{-2}}{(1-0.1q^{-1})(1-0.7q^{-1})(1-0.9q^{-1})}$$



Closed loop stable

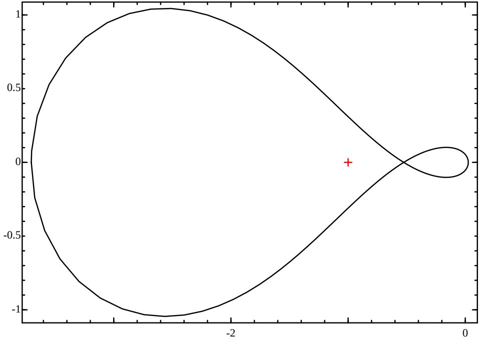
Gain margin

$0 < K < 3$

Example

Unstable open loop

$$G_{open}(\mathrm{q}^{-1}) = \frac{0.1\mathrm{q}^{-2}}{(1-0.1\mathrm{q}^{-1})(1-0.7\mathrm{q}^{-1})(1-1.1\mathrm{q}^{-1})}$$

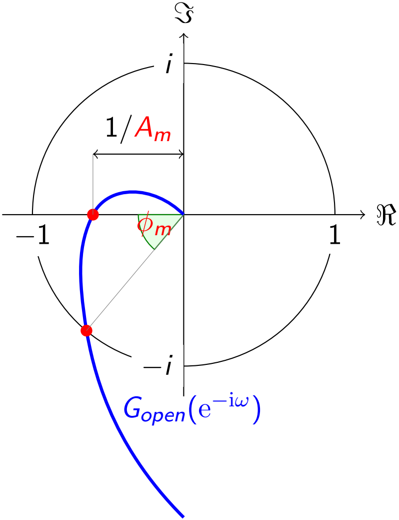


Closed loop stable

Gain margins

$$\frac{1}{3.7} = 0.27 < K < \frac{1}{0.5} = 2$$

Amplitude and phase margins



Nyquist curve in complex plane

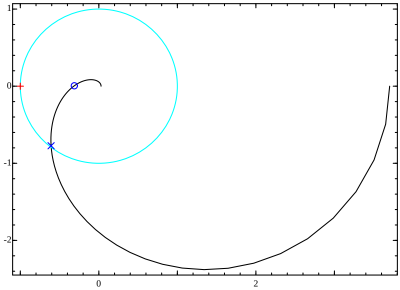
$$G_{open}(\mathrm{e}^{-\mathrm{i}\omega}), \quad \omega = 0 \to \pi$$

Closed loop stability margins

Amplitude (gain) margin: A_m

Phase margin: ϕ_m

Example: previous stable system



Closed loop stable for perturbations:

Gain margin

$$G_{open}(e^{-i\omega_o}) = -\frac{1}{A_m} = -0.32$$
$$u(k) = -Ky(k),\ K < A_m = 3.1$$

Phase margin

$$|G_{open}(e^{-i\omega_x})| = 1,$$
$$\phi_m = \arg G_{open}(e^{-i\omega_x}) + \pi$$
Delay margin $\tau_m = \phi_m/\omega_x = 3.11$
$$u(k) = -y(k-\tau),\ \tau = 0, 1, 2, 3$$

Stability margins in closed-loop

Let G_{open} be stable and $u(k) = 0$

$$y(k) = G_{open}u(k) \to 0, \quad k \to \infty$$

Gain variation in feedback

$$u(k) = -Ky(k)$$

Stable for $0 \leq K < A_m$

$$y(k) \to 0, k \to \infty$$

$K = A_m$: Oscillation freq. ω_o

Time delay variation in feedback

$$u(k) = -y(k - \tau)$$

Stable for $\tau < \tau_m = \phi_m/\omega_x$

$$y(k) \to 0, k \to \infty$$

$\tau = \tau_m$: Oscillation freq. ω_x