B Transforms

B.1 The Laplace Transform

B.1.1 The Laplace transform of causal signals

In the table below $f(t) = 0$ for $t < 0$ (i.e. $f(t) \cdot u(t) = f(t)$).

1.	$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \mathcal{F}(s) e^{st} ds$	$\longleftrightarrow$	$\mathcal{F}(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$
2.	$\sum_\nu a_\nu f_\nu(t)$	$\longleftrightarrow$	$\sum_\nu a_\nu \mathcal{F}_\nu(s)$ Linearity
3.	$f(at)$	$\longleftrightarrow$	$\frac{1}{a}\ \mathcal{F}(\frac{s}{a})$ Scaling
4.	$\frac{1}{a}\ f(\frac{t}{a})$	$\longleftrightarrow$	$\mathcal{F}(as)\ a > 0$ Scaling
5.	$f(t - t_0);\ t \geq t_0$	$\longleftrightarrow$	$\mathcal{F}(s)\ e^{-st_0}$ Time shift
6.	$f(t) \cdot e^{-at}$	$\longleftrightarrow$	$\mathcal{F}(s + a)$ Frequency shift
7.	$\frac{d^n f}{dt^n}$	$\longleftrightarrow$	$s^n \mathcal{F}(s)$ Derivation
8.	$\int_{0-}^{t} f(\tau) d\tau$	$\longleftrightarrow$	$\frac{1}{s}\ \mathcal{F}(s)$ Integration
9.	$(-t)^n f(t)$	$\longleftrightarrow$	$\frac{d^n \mathcal{F}(s)}{ds^n}$ Derivation in frequency domain
10.	$\frac{f(t)}{t}$	$\longleftrightarrow$	$\int_s^{\infty} \mathcal{F}(z) dz$ Integration in frequency domain
11.	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot \mathcal{F}(s)$		Initial value theorem
12.	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot \mathcal{F}(s)$		End value theorem
13.	$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_1(t - \tau) f_2(\tau) d\tau$	$\longleftrightarrow$	$\mathcal{F}_1(s) \cdot \mathcal{F}_2(s)$ Convolution in time domain
14.	$f_1(t) \cdot f_2(t)$	$\longleftrightarrow$	$\frac{1}{2\pi j}\ \mathcal{F}_1(s) * \mathcal{F}_2(s) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \mathcal{F}_1(z) \cdot \mathcal{F}_2(s - z) \cdot dz$ Convolution in frequency domain
15.	$\int_{0-}^{\infty} f_1(t) \cdot f_2(t) dt = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \mathcal{F}_1(s) \cdot \mathcal{F}_2(-s) ds$		Parseval's relation

16. $\delta(t) \longleftrightarrow 1$

17. $\delta^n(t) \longleftrightarrow s^n$

18. $1 \longleftrightarrow \frac{1}{s}$

19. $\frac{1}{n!}\, t^n \longleftrightarrow \frac{1}{s^{n+1}}$

20. $e^{-\sigma_0 t} \longleftrightarrow \frac{1}{s+\sigma_0}$

21. $\frac{1}{(n-1)!}\, t^{n-1} e^{-\sigma_0 t} \longleftrightarrow \frac{1}{(s+\sigma_0)^n}$

22. $\sin \Omega_0 t \longleftrightarrow \frac{\Omega_0}{s^2+\Omega_0^2}$

23. $\cos \Omega_0 t \longleftrightarrow \frac{s}{s^2+\Omega_0^2}$

24. $t \cdot \sin \Omega_0 t \longleftrightarrow \frac{2\Omega_0 s}{(s^2+\Omega_0^2)^2}$

25. $t \cdot \cos \Omega_0 t \longleftrightarrow \frac{s^2-\Omega_0^2}{(s^2+\Omega_0^2)^2}$

26. $e^{-\sigma_0 t} \sin \Omega_0 t \longleftrightarrow \frac{\Omega_0}{(s+\sigma_0)^2+\Omega_0^2}$

27. $e^{-\sigma_0 t} \cos \Omega_0 t \longleftrightarrow \frac{s+\sigma_0}{(s+\sigma_0)^2+\Omega_0^2}$

28. $e^{-\sigma_0 t} \sin(\Omega_0 t + \phi) \longleftrightarrow \frac{(s+\sigma_0)\sin\phi + \Omega_0 \cos\phi}{(s+\sigma_0)^2+\Omega_0^2}$

B.1.2 One-sided Laplace transform of non-causal signals

Notation

$\mathcal{F}^+(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$ — Single sided Laplace transform, $f(t)$ not necessarily causal.

$\mathcal{F}(s) = \mathcal{F}^+(s)$ — For causal signals

Taking the derivative of $f(t)$ yields

$$\frac{d}{dt} f(t) \longleftrightarrow s \cdot \mathcal{F}^+(s) - f(0-) \qquad \text{Single derivation}$$

$$\frac{d^n}{dt} f(t) \longleftrightarrow s^n \mathcal{F}^+(s) - s^{n-1} f(0-)$$
$$-s^{n-2} f^{(1)}(0-) - \ldots f^{(n-1)}(0-) \; n \text{ derivations}$$

B.2 The Fourier Transform of a Continuous Time Signal

$$\Omega = 2\pi F$$

1.	$w(t) = \mathcal{F}^{-1}\{W(F)\} = \int_{-\infty}^{\infty} W(F) e^{j2\pi F t} dF$	$\longleftrightarrow$	$W(F) = \mathcal{F}\{w(t)\} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi F t} dt$
2.	$\sum_\nu a_\nu w_\nu(t)$	$\longleftrightarrow$	$\sum_\nu a_\nu W_\nu(F)$
3.	$w^*(-t)$	$\longleftrightarrow$	$W^*(F)$
4.	$W(t)$	$\longleftrightarrow$	$w(-F)$
5.	$w(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a \rvert} W\left(\frac{F}{a}\right)$
6.	$w(t - t_0)$	$\longleftrightarrow$	$W(F) \cdot e^{-j2\pi F t_0}$
7.	$w(t) \cdot e^{j2\pi F_0 t}$	$\longleftrightarrow$	$W(F - F_0)$
8.	$w^*(t)$	$\longleftrightarrow$	$W^*(-F)$
9.	$\frac{d^n w(t)}{dt^n}$	$\longleftrightarrow$	$(j2\pi F)^n \, W(F)$
10.	$\int_{-\infty}^{t} w(\tau) d\tau$	$\longleftrightarrow$	$\frac{1}{j2\pi F} \, W(F)$ if $W(F) = 0$ for $F = 0$
11.	$-j2\pi t \; w(t)$	$\longleftrightarrow$	$\frac{dw}{dF}$
12.	$w_1(t) * w_2(t)$	$\longleftrightarrow$	$W_1(F) \cdot W_2(F)$
13.	$w_1(t) \cdot w_2(t)$	$\longleftrightarrow$	$W_1(F) * W_2(F)$
14.	$\int_{-\infty}^{\infty} \lvert w(t) \rvert^2 dt = \int_{-\infty}^{\infty} \lvert W(F) \rvert^2 dF$		Parseval's relation
15.	$\int_{-\infty}^{\infty} w_1(t) \cdot w_2(t) dt = \int_{-\infty}^{\infty} W_1(F) \cdot W_2^*(F) dF$		$w_1(t), \; w_2(t)$ real
16.	$\delta(t)$	$\longleftrightarrow$	1
17.	1	$\longleftrightarrow$	$\delta(F)$
18.	$u(t)$	$\longleftrightarrow$	$\frac{1}{j2\pi F} + \frac{1}{2} \, \delta(F)$
19.	$e^{-at} u(t)$	$\longleftrightarrow$	$\frac{1}{a + j\Omega}$

20. $e^{-a|t|}$ $\longleftrightarrow$ $\frac{2a}{a^2+\Omega^2}$

21. $e^{j2\pi F_0 t}$ $\longleftrightarrow$ $\delta(F-F_0)$

22. $\sin 2\pi F_0 t$ $\longleftrightarrow$ $j\,\frac{1}{2}\,\{\delta(F+F_0)-\delta(F-F_0)\}$

23. $\sin 2\pi F_0 t \cdot u(t)$ $\longleftrightarrow$ $\frac{\Omega_0}{\Omega_0^2-\Omega^2}+j\,\frac{1}{4}\,\{\delta(F+F_0)-\delta(F-F_0)\}$

24. $\cos 2\pi F_0 t$ $\longleftrightarrow$ $\frac{1}{2}\,\{\delta(F+F_0)+\delta(F-F_0)\}$

25. $\cos 2\pi F_0 t \cdot u(t)$ $\longleftrightarrow$ $\frac{j\Omega}{\Omega_0^2-\Omega^2}+\frac{1}{4}\{\delta(F+F_0)+\delta(F-F_0)\}$

26. $\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-t^2/2\sigma^2}$ $\longleftrightarrow$ $e^{-(\Omega\sigma)^2/2}$

27. $e^{-at}\sin 2\pi F_0 t \cdot u(t)$ $\longleftrightarrow$ $\dfrac{\Omega_0}{(j\Omega+a)^2+(\Omega_0)^2}$

28. $e^{-a|t|}\sin 2\pi F_0|t|$ $\longleftrightarrow$ $\dfrac{2\Omega_0(\Omega_0^2+a^2-\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$

29. $e^{-at}\cos 2\pi F_0 t \cdot u(t)$ $\longleftrightarrow$ $\dfrac{j\Omega+a}{(j\Omega+a)^2+(\Omega_0)^2}$

30. $e^{-a|t|}\cos 2\pi F_0 t$ $\longleftrightarrow$ $\dfrac{2a(\Omega_0^2+a^2+\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$

31. $\mathrm{rect}(at)=\begin{cases}1 \text{ for } |t|<\frac{1}{2a}\\ 0 \text{ Else}\end{cases}$ $\longleftrightarrow$ $\frac{1}{a}\,\mathrm{sinc}(\frac{F}{a})\quad a>0$

32. $\mathrm{sinc}(at)=\frac{\sin(\pi at)}{\pi at}$ $\longleftrightarrow$ $\frac{1}{a}\,\mathrm{rect}(\frac{F}{a})\quad a>0$

33. $\mathrm{rep}_T(w(t))=\sum_{m=-\infty}^{\infty}w(t-mT)$ $\longleftrightarrow$ $\frac{1}{|T|}\,\mathrm{comb}_{1/T}(W(F))$

34. $|T|\mathrm{comb}_T(w(t))=$
$|T|\sum_{m=-\infty}^{\infty}w(mT)\delta(t-mT)$ $\longleftrightarrow$ $\mathrm{rep}_{1/T}(W(F))$

35. $\sum_{n=-\infty}^{\infty}c_n\delta(t-nT)$ $\longleftrightarrow$ $\sum_{n=-\infty}^{\infty}\frac{1}{T}\,c_n\delta(F-\frac{n}{T})=\sum c_n e^{-j2\pi nTF}$

B.3 The Z-transform

B.3.1 The Z-transform of causal signals

1.	$\mathcal{X}(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$	Transform
2.	$x(n) = Z^{-1}[\mathcal{X}(z)] = \frac{1}{2\pi j}\int_\Gamma \mathcal{X}(z)z^{n-1}dz$	Invers transform
3.	$\sum_\nu a_\nu x_\nu(n) \longleftrightarrow \sum_\nu a_\nu \mathcal{X}_\nu(z)$	Linearity
4.	$x(n-n_0) \longleftrightarrow z^{-n_0}\mathcal{X}(z)$	Shift (n_0 positive or negative integer)
5.	$nx(n) \longleftrightarrow -z\,\frac{d}{dz}\,\mathcal{X}(z)$	Multiplication with n
6.	$a^n x(n) \longleftrightarrow \mathcal{X}\left(\frac{z}{a}\right)$	Scaling
7.	$x(-n) \longleftrightarrow \mathcal{X}\left(\frac{1}{z}\right)$	Reflection of the time sequence
8.	$\left[\sum_{\ell=-\infty}^{n} x(\ell)\right] \longleftrightarrow \frac{z}{z-1}\,\mathcal{X}(z)$	Summation
9.	$x * y \longleftrightarrow \mathcal{X}(z)\cdot\mathcal{Y}(z)$	Convolution
10.	$x(n)\cdot y(n) \longleftrightarrow \frac{1}{2\pi j}\int_\Gamma \mathcal{Y}(\xi)\mathcal{X}\left(\frac{z}{\xi}\right)\xi^{-1}d\xi$	Product
11.	$x(0) = \lim_{z\to\infty}\mathcal{X}(z)$ (if the limit value exist)	Initial value theorem
12.	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\mathcal{X}(z)$ (if the unit cirle is included in the ROC)	End value theorem
13.	$\sum_{\ell=-\infty}^{\infty} x(\ell)y(\ell) = \frac{1}{2\pi j}\int_\Gamma x(z)y\left(\frac{1}{z}\right)z^{-1}dz$	Parseval's teorem for real time sequences
14.	$\sum_{\ell=-\infty}^{\infty} x^2(\ell) = \frac{1}{2\pi j}\int_\Gamma \mathcal{X}(z)\mathcal{X}(z^{-1})z^{-1}dz$	– " –

	Sequence		Transform
	$x(n)$	$\longleftrightarrow$	$\mathcal{X}(z)$
15.	$\delta(n)$	$\longleftrightarrow$	1
16.	$u(n)$	$\longleftrightarrow$	$\frac{1}{1-z^{-1}}$
17.	$nu(n)$	$\longleftrightarrow$	$\frac{z^{-1}}{(1-z^{-1})^2}$
18.	$\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{1-\alpha z^{-1}}$
19.	$(n+1)\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^2}$
20.	$\frac{(n+1)(n+2)\dots(n+r-1)}{(r-1)!}\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^r}$
21.	$\alpha^n \cos\beta n \; u(n)$	$\longleftrightarrow$	$\frac{1-z^{-1}\alpha\cos\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
22.	$\alpha^n \sin\beta n \; u(n)$	$\longleftrightarrow$	$\frac{z^{-1}\alpha\sin\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
23.	$\mathbf{F}^n u(n)$	$\longleftrightarrow$	$(\mathbf{I}-z^{-1}\mathbf{F})^{-1}$

B.3.2 Single-Sided Z-transform of non-causal signals

Notation

$\mathcal{X}^+(z)=\sum_{n=0}^{\infty} x(n)z^{-n}$	Single-sided z-transform, $x(n)$ not necessarily causal.
$\mathcal{X}(z)=\mathcal{X}^+(z)$	For causal signals

Shifting $x(n)$yields:
i) one step shift

$$x(n-1) \longleftrightarrow z^{-1}\mathcal{X}^+(z)+x(-1)$$
$$x(n+1) \longleftrightarrow z\mathcal{X}^+(z)-x(0)\cdot z$$

ii) n_0 step shift ($n_0 \geq 0$)

$$x(n-n_0) \longleftrightarrow z^{-n_0}\mathcal{X}^+(z)+x(-1)z^{-n_0+1}+ +x(-2)z^{-n_0+2}+\dots+x(-n_0)$$
$$x(n+n_0) \longleftrightarrow z^{n_0}\mathcal{X}^+(z)-x(0)z^{n_0}-x(1)z^{n_0-1}-\dots-x(n_0-1)z$$

B.4 Fourier Transform for Discrete Time Signal

1.	$X(f) = \mathcal{F}(x(n)) =$ $= \sum_{\ell=-\infty}^{\infty} x(\ell) e^{-j2\pi f \ell} \quad \omega = 2\pi f$		Transform
2.	$x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df =$ $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(f) e^{j\omega n} d\omega$		Inverse transform
3.	$\sum a_\nu x_\nu(n)$	$\longleftrightarrow$	$\sum_\nu a_\nu X_\nu(f)$ Linearity
4.	$x(n - n_0)$	$\longleftrightarrow$	$X(f) \cdot e^{-j2\pi f n_0}$ Shift
5.	$x(n) e^{j2\pi f_0 n}$	$\longleftrightarrow$	$X(f - f_0)$ Frequency translation
6.	$x(n) \cdot \cos 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$ Modulation
7.	$x(n) \cdot \sin 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2j}\left[X(f - f_0) - X(f + f_0)\right]$ Modulation
8.	$x * y$	$\longleftrightarrow$	$X(f) \cdot Y(f)$ Convolution
9.	$x \cdot y$	$\longleftrightarrow$	$\int_{-1/2}^{1/2} X(\lambda) \cdot Y(f - \lambda) d\lambda$ Product
10.	$\sum_{\ell=-\infty}^{\infty} x(\ell) y(\ell) =$ $= \int_{-1/2}^{1/2} X(f) Y^*(f) df$		Parseval's teorem for real time sequences
11.	$X(f) = \mathcal{X}(e^{j\omega})$		if $x(n) = 0$ for $n < n_0$ and $\sum_{\ell=-\infty}^{\infty} \lvert x(\ell) \rvert^2 < \infty$ (Valid for for example: 18,19,20,21 och 22 in the Z-transform table for $\lvert\alpha\rvert < 1$)
12.	$\delta(n)$	$\longleftrightarrow$	1
13.	$\delta(n - n_0)$	$\longleftrightarrow$	$e^{-j\omega n_0}$
14.	$1 \ \forall n$	$\longleftrightarrow$	$\sum_{p=-\infty}^{\infty} \delta(f - p)$
15.	$u(n)$	$\longleftrightarrow$	$\frac{1}{2} \sum_{p=-\infty}^{\infty} \delta(f - p) + \frac{1}{2} + \frac{1}{j \cdot 2 \cdot \tan(\pi f)}$

16. $2f_1 \cdot \mathrm{sinc}(2f_1 \cdot n) = 2f_1 \, \dfrac{\sin(2\pi f_1 n)}{2\pi f_1 n}$

$$\longleftrightarrow \mathrm{rect}_p\left(\frac{f}{2f_1}\right) = \begin{cases} 1 & |f-n| < f_1 < 1/2 \ , \ n \text{ integer} \\ 0 & \text{Else} \end{cases}$$

Ideal LP filter

17. $4f_1 \mathrm{sinc}(2f_1 n)\cos(2\pi f_0 n)$

$$\longleftrightarrow \mathrm{rect}_p\left(\frac{f-f_0}{2f_1}\right) + \mathrm{rect}_p\left(\frac{f+f_0}{2f_1}\right) \text{ Ideal BP Filter}$$

18. $\dfrac{2\pi f_1 n \cos 2\pi f_1 n - \sin 2\pi f_1 n}{\pi n^2}$

$$\longleftrightarrow (j2\pi f)_p = \begin{cases} j2\pi(f-n) & |f-n| < f_1 < 1/2 \ , \ n \text{ integer} \\ 0 & \text{Else} \end{cases}$$

"Derivating" system

19. $\cos(2\pi f_0 n) \longleftrightarrow \dfrac{1}{2}\sum_{p=-\infty}^{\infty}[\delta(f-f_0-p)+\delta(f+f_0-p)]$

20. $\alpha^{|n|} \longleftrightarrow \dfrac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f}$

21. $\alpha^{|n|}\cos(2\pi f_0 n)$

$$\longleftrightarrow \frac{1-\alpha^2}{2}\left[\frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f+f_0)} + \frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f-f_0)}\right]$$

22. $p_r(n) = \begin{cases} 1 & |n| \leq \dfrac{M-1}{2} \\ 0 & \text{Else} \end{cases} \quad M \text{ odd}$

$$\longleftrightarrow P_r(f) = \frac{\sin(\pi f M)}{\sin(\pi f)} \text{ Rectangular window}$$

B.5 Some DFT Properties

Time	Frequency
$x(n),\ y(n)$	$X(k),\ Y(k)$
$x(n) = x(n+N)$	$X(k) = X(k+N)$
$x(N-1)$	$X(N-k)$
$x((n-1))_{<N>}$	$X(k)e^{-j2\pi k1/N}$
$x(n)e^{j2\pi 1n/N}$	$X((k-1))_{<N>}$
$x^*(n)$	$X^*(N-k)$
$x_1(n) \circledast x_2(n)$	$X_1(k)X_2(k)$
$x(n) \circledast y^*(-n)$	$X(k)Y^*(k)$
$x_1(n)x_2(n)$	$\frac{1}{N}\ X_1(k)\ N\ X_2(k)$
$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$