Reference formulas and equations in Multivariable Calculus

Trigonometry and Logarithms

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\pm \cot x + \cot y}$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$\sin 2x = 2 \sin x \cos x$$

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$2 \sin x \cos y = \sin(x - y) + \sin(x + y)$$

$$\ln x + \ln y = \ln xy$$

$$\ln x - \ln y = \ln \frac{x}{y}$$

$$\ln x^a = a \ln x$$

$$(x, y > 0)$$

Standard limits

$$\lim_{x \to 0^+} x^\alpha \log_a x = 0 \ (a > 1,\ \alpha > 0)$$

$$\lim_{x \to \infty} \frac{a^x}{x^\alpha} = \infty \ (a > 1)$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to \infty} \frac{x^\alpha}{\log_a x} = \infty \ (a > 1,\ \alpha > 0)$$

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Basic derivatives

$f(x)$	$f'(x)$
x^a	ax^{a-1}
a^x	$a^x \ln a$
$\ln \lvert x \rvert$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1 - x^2}}$
$\arctan x$	$\frac{1}{1 + x^2}$
$\ln \left\lvert x + \sqrt{x^2 + \alpha} \right\rvert$	$\frac{1}{\sqrt{x^2 + \alpha}}$
$\frac{1}{2} x \sqrt{x^2 + \alpha} + \frac{\alpha}{2} \ln \left\lvert x + \sqrt{x^2 + \alpha} \right\rvert$	$\sqrt{x^2 + \alpha}$

Taylor Series

Taylor's formula for a function $f: \mathbb{R} \to \mathbb{R}$

$$f(a+h) = f(a) + f'(a)\,h + \frac{f''(a)}{2}\,h^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,h^n$$

Table of particular expansions ($a = 0,\ h \to x$)

1. $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$

2. $(x+1)^\alpha = 1 + \alpha\,x + \dfrac{\alpha(\alpha-1)}{2}\,x^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{2\cdot 3}\,x^3 + \cdots \quad (-1 < x < 1)$

3. $e^x = \sum_{k=0}^{\infty} \dfrac{1}{k!}\,x^k = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \cdots$

4. $\sin x = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{(2k-1)!}\,x^{2k-1} = x - \dfrac{1}{6}x^3 + \dfrac{1}{120}x^5 - \cdots$

5. $\cos x = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!}\,x^{2k} = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \cdots$

6. $\ln(x+1) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}\,x^k = x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 - \cdots \quad (-1 < x \le 1)$

7. $\arctan x = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{(2k-1)}\,x^{2k-1} = x - \dfrac{1}{3}x^3 + \dfrac{1}{5}x^5 - \cdots \quad (-1 \le x \le 1)$

Taylor's formula for a function $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(a+h, b+k) = f(a,b) + h\,f_x(a,b) + k f_y(a,b) + \frac{1}{2}\Big(h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\Big) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)_{(a,b)}$$

Tangent plane

Function $z = f(x,y)$

Equation of tangent plane through the point $(a, b, f(a,b))$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Level surface $F(x,y,z) = C$

Equation of tangent plane through the point (a,b,c)

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

Directional derivative

The directional derivative of a function $f: \mathbb{R}^3 \to \mathbb{R}$ at the point (a,b,c) and direction $\boldsymbol{u}$ $(|\boldsymbol{u}| = 1)$

$$D_{\boldsymbol{u}} f(a,b,c) = f'_{\boldsymbol{u}}(a,b,c) = \boldsymbol{u}\cdot\nabla f(a,b,c) = \boldsymbol{u}\cdot(f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)).$$

Double Integrals

General substitution

Assume a one-to-one mapping between a region D in the xy-plane and a region D' in the uv-plane

$$\begin{cases} x &= x(u,v) \\ y &= y(u,v) \end{cases} \Leftrightarrow \begin{cases} u &= u(x,y) \\ v &= v(x,y) \end{cases}$$

Then $\displaystyle\iint_D f(x,y)\,dxdy = \iint_{D'} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$

with $\displaystyle\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0.$

Polar coordinates

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r, \quad \iint_D f(x,y)\,dxdy = \iint_{D'} f(r\cos\theta, r\sin\theta)\,rdr\,d\theta$$

Triple Integrals

General substitution

As above assume a one-to-one mapping between points (x,y,z) in Δ and (u,v,w) in Δ'.

$$\iint_\Delta f(x,y,z)\,dxdydz = \iint_{\Delta'} f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

with $\displaystyle\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

Spherical coordinates

$$\begin{cases} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$$

$$\iiint_\Delta f(x,y,z)\,dxdydz = \iiint_{\Delta'} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\,\sin\phi\,d\rho\,d\phi\,d\theta$$

Line Integrals

Tangent line integral

Given a parametrized curve $C: \mathbf{r}(t) = (x(t), y(t), z(t)),\ a \le t \le b$ and a vector field $\boldsymbol{F} = (P, Q, R)$.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b (P, Q, R) \cdot (x'(t), y'(t), z'(t))\, dt = \int_a^b \Big(P\, x'(t) + Q\, y'(t) + R\, z'(t)\Big)\, dt$$

Line integral with respect to arc length

Assume a curve C as above and a function $f: \mathbb{R}^3 \to \mathbb{R}$.

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt, \quad (ds = d|\boldsymbol{r}|)$$

Green's theorem

Given a plane, closed, positively oriented curve C that encloses a region D and a field $\boldsymbol{F} = (P, Q)$.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dxdy$$

Surface Integrals

General parametrized surface

$S: \boldsymbol{r} = \boldsymbol{r}(u, v) = (x(u, v), y(u, v), z(u, v)),\ (u, v) \epsilon D.$

$$\iint_S f(x, y, z)\, dS = \iint_D f(\boldsymbol{r}(u, v)) \Big|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\Big|\, dudv$$

Function graph $z = h(x, y)$

$S: \boldsymbol{r} = (x, y, z) = (x, y, h(x, y)).$

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + h_x^2 + h_y^2}\, dxdy$$