501. $\{X(t)\}$ $\quad m_X = 0, \quad r(\tau) = \frac{1}{1+\tau^2}$ (svagt stationär)

$$\int_0^1 X(t)\,dt = \int_0^1 \underbrace{1}_{g(t)} \cdot X(t)\,dt$$

där $g(t)$ begr. och R-integrerbar på $(0,1)$ så enl. sats 1 (s. 106)

$$E\left(\int_0^1 X(t)\,dt\right) = m_X \cdot \int_0^1 1 \cdot dt = \underline{\underline{0}}$$

$$V\left(\int_0^1 X(t)\,dt\right) = C\left(\int_0^1 X(s)\,ds, \int_0^1 X(t)\,dt\right)$$

$$= \int_0^1\int_0^1 r(s-t)\,dt\,ds$$

$$= \int_0^1\int_0^1 \frac{1}{1+(s-t)^2}\,dt\,ds$$

$$= \int_0^1\left(\int_0^1 \frac{1}{1+(s-t)^2}\,ds\right)dt$$

$\left\{\begin{array}{ll} u = s-t & s=1 \Leftrightarrow u=1-t \\ du = ds & s=0 \Leftrightarrow u=-t \end{array}\right\}$

$$= \int_0^1\left(\int_{-t}^{1-t} \frac{1}{1+u^2}\,du\right)dt$$

$$= \int_0^1 \left[\arctan u\right]_{-t}^{1-t}\,dt$$

$\left\{\begin{array}{ll} v = 1-t & t=1 \Leftrightarrow v=0 \\ dv = -dt & t=0 \Leftrightarrow v=1 \end{array}\right\}$

$$= \int_0^1 \left(\arctan(1-t) - \arctan(-t)\right)dt$$

$$= \int_1^0 \arctan(v)\,(-dv) - \int_0^1 \arctan(-t)\,dt$$

$$= 2\int_0^1 \arctan(t)\,dt$$

$$\int fg = Fg - \int Fg'$$

$$= 2\int_0^1 \underbrace{1}_{f} \cdot \underbrace{\arctan}_{g} t \, dt$$

$$= 2\left([t \arctan t]_0^1 - \int_0^1 t \frac{1}{1+t^2} dt \right) \qquad \left\{ \begin{matrix} u = t^2 \\ du = 2t\,dt \end{matrix} \right\}$$

$$= 2\left(\arctan 1 - \int_0^1 \frac{\frac{1}{2} du}{1+u} \right)$$

$$= 2\left(\arctan 1 - \frac{1}{2}[\ln(1+u)]_0^1 \right)$$

$$= 2 \arctan 1 - (\ln(2) - \ln(1))$$

$$= 2 \arctan 1 - \ln 2 \approx 0.87765$$

Om $X(t)$ normal-process så är $\int_0^1 X(t)\,dt \in N(0, \sqrt{0.87765})$

så $P\left(\int_0^1 X(t) < -1\right) = \Phi\left(\frac{-1-0}{\sqrt{0.87765}}\right) =$

$$= 1 - \underbrace{\Phi\Big(\underbrace{\frac{1}{\sqrt{0.87765}}}_{\approx 1.07}\Big)}_{0.86} = 0.14$$

502) $\{X(t)\}$ $m_X = 0$ $r_X = \max(0, 1-|\tau|)$ (s. 101)

$$Y(t) = \int_{t-2}^{t} X(u)\,du$$

a) Vad är impulssvaret $h(v)$:

$$Y(t) = \int_{-\infty}^{\infty} h(v) X(t-v)\,dv \ ?$$

$$Y(t) = \int_{t-2}^{t} X(u)\,du \qquad \left\{\begin{array}{l|l} v = t-u & u=t \leftrightarrow v=0 \\ dv = -du & u=t-2 \leftrightarrow v=2 \end{array}\right\}$$

$$= \int_{2}^{0} X(t-v)\cdot(-dv)$$

$$= \int_{0}^{2} X(t-v)\,dv$$

$$= \int_{-\infty}^{\infty} h(v) X(t-v)\,dv \quad \text{där } h(v) = \begin{cases} 1 & 0<v<2 \\ 0 & \text{annars} \end{cases}$$

varmed $h(v) = 0$ då $v<0$ så <u>kausalt!</u>

b) Filtrets frekv. fkn: $H(f) = \int_{-\infty}^{\infty} e^{-i2\pi f u} h(u)\,du$

$$H(f) = \int_{-\infty}^{\infty} e^{-i2\pi f u} h(u)\,du$$

$$= \int_{0}^{2} e^{-i2\pi f u}\cdot 1\,du \qquad \Bigg| \quad = \int_{0}^{2} 1\cdot du$$

$$= \left[-\frac{e^{-i2\pi f u}}{i2\pi f}\right]_0^2 \text{ då } f \neq 0 \qquad \Bigg| \quad = 2 \text{ då } f=0$$

$$= -\frac{e^{-i4\pi f} - 1}{i2\pi f}$$

$$H(f) = \begin{cases} \dfrac{1-e^{-i4\pi f}}{i2\pi f} & \text{då } f \neq 0 \\ 2 & \text{då } f=0 \end{cases}$$

Sats 3: (s. 110)

c) Utsignalens spektr.täthet $R_Y = |H|^2 R_X$

$$r_X(\tau) = \max(0, 1-|\tau|)$$

$$= \begin{cases} 1-|\tau| & \text{om } |\tau| \le 1 \\ 0 & \text{annars} \end{cases}$$

$$\Rightarrow \quad R_X(f) = \begin{cases} \frac{2}{(2\pi f)^2}(1-\cos(2\pi f)) & f \neq 0 \\ 1 & f = 0 \end{cases}$$

$f = 0$:

$$R_Y(f) = |H(f)|^2 \cdot 1$$

$$= \left| \frac{1-e^{-i4\pi f}}{i2\pi f} \right|^2 \qquad |z|^2 = z \cdot \bar{z}$$

$$= \frac{(1-e^{-i4\pi f})(1-e^{i4\pi f})}{|i2\pi f|^2}$$

$$= \frac{1-e^{-i4\pi f} - e^{i4\pi f} + e^{-i4\pi f + i4\pi f}}{(2\pi f)^2}$$

$$= \frac{2\left(1 - \frac{e^{-i4\pi f} + e^{i4\pi f}}{2}\right)}{(2\pi f)^2}$$

$$= \frac{2(1-\cos(4\pi f))}{(2\pi f)^2} = \frac{1-\cos(4\pi f)}{2(\pi f)^2}$$

$f \neq 0$:

$$R_Y(f) = |H(f)|^2 \cdot \frac{2}{(2\pi f)^2}(1-\cos(2\pi f))$$

$$= \frac{2(1-\cos(4\pi f))}{(2\pi f)^2} \cdot \frac{2}{(2\pi f)^2}(1-\cos(2\pi f))$$

$$= \frac{4(1-\cos(4\pi f))(1-\cos(2\pi f))}{(2\pi f)^4}$$

$4(1-\cos(4\pi f)$

503. a) Medeleffekten $E(Y(t)^2)$

$$m_Y = m_X \int_{\mathbb{R}} h(u)\,du \quad \text{(Sats 2, s. 108)}$$

$$= m_X \int e^{-i2\pi\cdot(0)\cdot u} h(u)\,du$$

$$= m_X \underbrace{H(0)}_{=0 \text{ enl. figur}}$$

$$= 0$$

$$R_Y = |H|^2 R_X$$

$$= R_X(f) \quad \text{då } f_0 - \frac{\Delta f}{2} \le |f| \le f_0 + \frac{\Delta f}{2}$$

så medeleffekten blir

$$E(Y(t)^2) = r_Y(0)$$

$$= \int_{-\infty}^{\infty} R_Y(f)\,df$$

$$= \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} R_X(f)\,df + \int_{-f_0 - \frac{\Delta f}{2}}^{-f_0 + \frac{\Delta f}{2}} R_X(f)\,df$$

$$= 2 \int_{f_0 - \Delta f/2}^{f_0 + \Delta f/2} R_X(f)\,df$$

b) om $\{X(t)\}$ har kont.

spektrum och R_X kont. then ...

skatta $R_X(f_0)$!

$$R_X(f_0) \approx \frac{1}{\Delta f} \int_{f_0-\Delta f/2}^{f_0+\Delta f/2} R_X(f)\,df$$

$$= \frac{1}{2\Delta f} E(Y(t)^2)$$

dvs $(a_0)X(t) + a_1 X(t-1) + a_2 X(t-2) = e(t)$

4. $X(t) = e(t) - 0.25\,X(t-2)$ (AR(2)-pr.)

där $E(e(t)) = 0$ och $V(e(t)) = 15$

och $\{e(t)\}$ ober. !

Yule-Walker-ekv.

kvf $r(\tau) + a_1 r(\tau-1) + a_2 r(\tau-2) = 0$ då $t = 1, 2, \ldots$

och (B.V. :) $r(0) + a_1 r(1) + a_2 r(2) = \sigma^2 = 15$

där $a_1 = 0$ och $a_2 = 0.25$ varmed

$$\begin{cases} r(\tau) + 0.25\,r(\tau-2) = 0 & t = 1, 2, \ldots \quad (1) \\ r(0) + 0.25\,r(2) = 15 & (2) \end{cases}$$

speciellt med $\tau = 2$: $r(2) + 0.25\,r(0) = 0$ (1')

dvs $4(1') - (2)$: $4r(2) - 0.25\,r(2) = -15$

$r(2) = -15/3.75 = -4$

$(1') \Rightarrow r(0) = 16$

($r(1) = 0$ eftersom $a_1 = 0$. Man kan också se detta genom att (1) då $\tau = 1$ ger $r(1) + 0.25\underbrace{r(-1)}_{=r(1)} = 0$ dvs $1.25\,r(1) = 0$)

Eftersom (1) måste $r(3) = -0.25\,r(1) = 0$ och på samma sätt med varje udda diff $\tau = 2k+1$: $r(\tau) = 0$.

Vidare ger (1) att $r(4) = -0.25\,r(2) = 1$

$r(6) = -0.25\,r(4) = -0.25$

osv.

varmed för varje jämn diff $\tau = 2k \geq 2$:

$r(\tau) = (-0.25)^{\tau}\, r(0) = 16 \cdot (-0.25)^{\tau}$

Eftersom $r(\tau)$ jämn ($C(X,Y) = C(Y,X)$) så är $r(\tau) = 16\,(-0.25)^{|\tau|}$ och alltså

$$r(\tau) = \begin{cases} 16 \cdot (-0.25)^{|\tau|} & \text{då } \tau = 2k \\ 0 & \text{då } \tau = 2k+1 \end{cases}$$

$\underline{R_f}$ – $X(t) = e(t) - 0.25\, X(t-2)$

$a_0 X(t) + a_1 X(t-1) + a_2 X(t-2) = e(t)$

där $a_0 = 1$, $a_1 = 0$, $a_2 = 0.25$

Enl. sats 4 (s. 115) är

$$R(f) = \frac{\sigma^2}{\left|\sum_{k=0}^{p} a_k e^{-i2\pi f k}\right|^2}$$

$$= \frac{15}{\left|1 \cdot e^{-i2\pi f \cdot 0} + 0.25\, e^{-i2\pi f \cdot 2}\right|^2}$$

$$= \frac{15}{\left|1 + 0.25\, e^{-i4\pi f}\right|^2}$$

$$= \frac{15}{(1 + \frac{1}{4} e^{-i4\pi f})(1 + \frac{1}{4} e^{i4\pi f})}$$

$$= \frac{15}{1 + \frac{1}{2}\left(\frac{e^{-i4\pi f} + e^{i4\pi f}}{2}\right) + \frac{1}{16}}$$

$$= \frac{15}{\frac{17}{16} + \frac{1}{2}\cos(4\pi f)} \quad \left\{\begin{matrix}\text{förlänger} \\ \text{med } 16\end{matrix}\right\}$$

$$= \frac{240}{17 + 8\cos(4\pi f)}$$

$-\frac{1}{2} < f \leq \frac{1}{2}$

(ty "samplad" dvs endast def:ad i heltal)

510. $X_t = e_t + 2e_{t-1} - e_{t-3}$

$E(e_t) = m$, $V(e_t) = \sigma^2$

vvf $E(X_t) = E(e_t) + 2E(e_{t-1}) - E(e_{t-3}) = 2m$

dvs $Y_t = X_t - 2m$ är en MA(3).

kvf Eftersom $r_X(s,t) = C(X(s), X(t)) =$

$= C(X(s) - 2m, X(t) - 2m)$

$= r_Y(s,t)$ kan kvf beräknas

enl. sats 5 (s. 121)

$$r_X(\tau) = \begin{cases} \sigma^2 \sum\limits_{s-t=\tau} c_s c_t & |\tau| \leq q \\ 0 & \text{annars} \end{cases}$$

$$= \begin{cases} \sigma^2(c_0c_0 + c_1c_1 + c_2c_2 + c_3c_3) & \tau = 0 \\ \sigma^2(c_1c_0 + c_2c_1 + c_3c_2) & \tau = \pm 1 \\ \sigma^2(c_2c_0 + c_3c_1) & \tau = \pm 2 \\ \sigma^2(c_3c_0) & \tau = \pm 3 \\ 0 & |\tau| \geq 4 \end{cases}$$

$$= \begin{cases} \sigma^2(1 + 4 + 0 + 1) = 6\sigma^2 & \tau = 0 \\ \sigma^2(2 + 0 + 0) = 2\sigma^2 & \tau = \pm 1 \\ \sigma^2(0 + (-2)) = -2\sigma^2 & \tau = \pm 2 \\ \sigma^2(-1) = -\sigma^2 & \tau = \pm 3 \\ 0 & |\tau| \geq 4 \end{cases}$$

R_f $R(f) = r(0) + 2\sum\limits_{\tau=1}^{3} r(\tau)\cos(2\pi f \tau)$

$= 6\sigma^2 + 2\left(2\sigma^2\cos(2\pi f) - 2\sigma^2\cos(2\pi f \cdot 2) - \sigma^2\cos(2\pi f \cdot 3)\right)$

$= 2\sigma^2\left(3 + 2\cos(2\pi f) - 2\cos(4\pi f) - \cos(6\pi f)\right)$

(6.) $X(t) - 1.559X(t-1) + 0.81X(t-2) = 3.5 + e(t)$

$E(e(t)) = 0 \quad V(e(t)) = 4$

<u>vvf</u> Antag att $E(X(t)) = m$. Då är

$$E(\text{V.L.}) = E(X(t)) - 1.559E(X(t-1)) + 0.81E(X(t-2))$$
$$= m - 1.559m + 0.81m$$
$$= 0.251m = 3.5 - 0 = E(\text{H.L.})$$

dvs $m \approx 13.944$

<u>kvf</u> Låt nu $Y(t) = X(t) - m$. Då gäller

$$Y(t) - 1.559Y(t-1) + 0.81Y(t-2) = e(t)$$

varmed $\{Y(t)\}$ är en <u>AR(2)-process</u>

med $a_1 = -1.559$ och $a_2 = 0.81$

Eftersom $r_X = r_Y$ fås av Yule-Walker ekv.

$$\begin{cases} r(\tau) - 1.559r(\tau-1) + 0.81r(\tau-2) = 0 & \tau = 1,2,\ldots \\ r(0) - \underbrace{1.559}_{a}r(1) + \underbrace{0.81}_{b}r(2) = 4 \end{cases}$$

$$\begin{cases} r(1) - a\,r(0) + b\,r(-1) = 0 & (1) \\ r(2) - a\,r(1) + b\,r(0) = 0 & (2) \\ r(0) - a\,r(1) + b\,r(2) = 4 & (3) \end{cases}$$

$$\begin{cases} b\,r(0) - a\,r(1) + r(2) = 0 & (2') \\ r(0) - a\,r(1) + b\,r(2) = 4 & (3') \\ -a\,r(0) + (b+1)r(1) = 0 & (1') \end{cases}$$

$$\begin{cases} b\,r(0) - a\,r(1) + r(2) = 0 & (2') \\ (b^2-1)r(0) - a(b-1)r(1) = -4 & (3'') = b(2') - (3') \\ -a\,r(0) + (b+1)r(1) = 0 & (1') \end{cases}$$

$$\begin{cases} b\,r(0) - a\,r(1) + r(2) = 0 & (2') \\ (b^2-1)\,r(0) - a(b-1)\,r(1) = -4 & (3'') \\ [(b+1)(b^2-1) - a^2(b-1)]\,r(0) = -4(b+1) & (1'') = (b+1)(3'') + a(b-1)(1') \end{cases}$$

$$\begin{cases} b\,r(0) - a\,r(1) + r(2) = 0 & (2') \\ (b^2-1)\,r(0) - a(b-1)\,r(1) = -4 & (3'') \\ r(0) = \dfrac{-4(b+1)}{\underbrace{(b+1)(b^2-1) - a^2(b-1)}_{45.061976}} \end{cases}$$

$$r(2) = a\,r(1) - b\,r(0) = 24.0093445 \approx \underline{\underline{24.01}}$$

$$r(1) = \frac{4 + (b^2-1)\,r(0)}{a(b-1)} = 38.81305 \approx \underline{\underline{38.81}}$$

$$r(0) = \underline{\underline{45.06}} = V(X(t))$$

$\underline{R_f}$

$$R(f) = \frac{\sigma^2}{\left|\sum_{k=0}^{2} a_k e^{-i2\pi f k}\right|^2}$$

$$= \frac{4}{\left|1 - 1.559e^{-i2\pi f} + 0.81e^{-i4\pi f}\right|^2}$$

$$= \frac{4}{(1 - 1.559e^{-i2\pi f} + 0.81e^{-i4\pi f})(1 - 1.559e^{i2\pi f} + 0.81e^{i4\pi f})}$$

$$= \frac{4}{\underline{1} + \underline{1.559^2} + \underline{0.81^2} - \underline{1.559e^{i2\pi f}} + 0.81e^{i4\pi f} - 1.559e^{-i2\pi f} - \underline{1.559 \cdot 0.81e^{i2\pi f}} + 0.81e^{-i4\pi f} - 0.81 \cdot 1.559e^{-i2\pi f}}$$

$$= \frac{4}{4.09 - 2.82(e^{i2\pi f} + e^{-i2\pi f}) + 0.81(e^{i4\pi f} + e^{-i4\pi f})}$$

$$= 4/(4.09 - 5.64\cos 2\pi f + 1.62\cos 4\pi f)$$

DERIVATA – PROCESSEN

520. $\{X(t)\}$ stationär normalpr. $r_X(\tau) = e^{-\tau^2/2}$, $m_X = 0$

a) $r_{X'}(\tau) = -r_X''(\tau)$ (enl. sats 9, s. 127)

$$r_X'(\tau) = \left(e^{-\tau^2/2}\right)'$$

$$= -\frac{2\tau}{2} e^{-\tau^2/2}$$

$$= -\tau e^{-\tau^2/2}$$

$$r_X''(\tau) = -1 \cdot e^{-\tau^2/2} - \tau\left(-\frac{2\tau}{2} e^{-\tau^2/2}\right)$$

$$= -e^{-\tau^2/2}\left(1 - \tau^2 e^{-\tau^2/2}\right)$$

så $r_{X'}(\tau) = e^{-\tau^2/2}\left(1 - \tau^2 e^{-\tau^2/2}\right)$
(jämn fkn)

b) $r_{X,X'}(\tau) = r_X'(\tau)$ (enl. sats 10, s. 129)

$= -\tau e^{-\tau^2/2}$ (från föreg. uppg.)
(udda fkn !!)

c) "Felet": $X(t+0.5) - X(t) - 0.5X'(t) =: Y(t)$

Då är $E\left(\left(X(t+0.5) - X(t) - 0.5X'(t)\right)^2\right) =$

$= \{ m_Y = 0$ ty $m_X = 0$ och $m_{X'} = 0$ enl. sats 9 $\} =$

* $= C\left(X(t+0.5) - X(t) - 0.5X'(t),\ X(t+0.5) - X(t) - 0.5X'(t)\right)$

$= V(X(t+0.5)) + V(X(t)) + 0.25\,V(X'(t)) -$

$- 2C(X(t+0.5), X(t)) - 0.5\,C(X(t+0.5), X'(t)) +$

$+ 2 \cdot 0.5\,C(X(t), X'(t)) - 0.5\,C(X'(t), X(t+0.5))$

$= 2r_X(0) + 0.25\,r_{X'}(0) - 2r_X(0.5) - 0.5\,r_{X,X'}(0.5) +$

$+ r_{X,X'}(0) - \underbrace{0.5\,r_{X',X}(-0.5)}_{= 0.5\,r_{X,X'}(0.5) \text{ ty udda fkn}}$

$$= 2e^{0} + 0.25e^{0}(1 - 0^{2}\cdot e^{0}) - 2e^{-0.125} -$$

$$- 0.5(-(-0.5)e^{-0.125}) + 0\cdot e^{0} - 0.5e^{-0.125}$$

$$= 2 + 0.25 - 2.5e^{-0.125}$$

$$\approx 0.04358$$

Eftersom $\{X(t)\}$ var normal process så är felet $Y(t)$ en normalfördelad variabel med $E(Y(t)) = 0$ (se $*$) och $V(Y(t)) \approx 0.044$ varmed

$$P(|Y(t)| > \tfrac{1}{4}) =$$

$$= 1 - P(-\tfrac{1}{4} < Y(t) < \tfrac{1}{4})$$

$$= 1 - \left(P(Y(t) \leq \tfrac{1}{4}) - P(Y(t) \leq -\tfrac{1}{4})\right)$$

$$= 1 - \left(\Phi\left(\frac{0.25}{\sqrt{0.04358}}\right) - \Phi\left(\frac{-0.25}{0.209}\right)\right)$$

$$= 1 - \left(\Phi\left(\frac{0.25}{0.209}\right) - \left(1 - \Phi\left(\frac{0.25}{0.209}\right)\right)\right)$$

$$= 2 - 2\underbrace{\Phi\Big(\underbrace{\frac{0.25}{0.209}}_{1.19}\Big)}_{0.883}$$

$$= \underline{\underline{0.234}}$$