601. $m_n^* = \frac{1}{n}\sum_{t=1}^{n} x(t)$ ($\bar{x}$ helt enkelt)

$= \frac{1}{12} 20.6$ (Sats 2, s. 258)

$= 1.71666\ldots$

$r_n(\tau)^* = \frac{1}{n}\sum_{t=1}^{n-\tau}(x(t)-\bar{x})(x(t+\tau)-\bar{x})$ (s. 164)

$$r(0)^* = \frac{1}{12}\sum_{1}^{12}(x(t)-\bar{x})^2$$

$$= \frac{1}{12}\sum_{1}^{12} x(t)^2 - \underbrace{\bar{x}^2}_{=5.87/(3\cdot 12)}$$

$$= \frac{1}{12} 37.32 - 2.9469444\ldots$$

$$= 0.16305555\ldots$$

$$r(2)^* = \frac{1}{12}\sum_{t=1}^{12-2}(x(t)-\bar{x})(x(t+2)-\bar{x})$$

$$= \frac{1}{12}\left(\sum_{1}^{10} x(t)x(t+2) - \bar{x}\sum_{1}^{10} x(t) - \bar{x}\sum_{1}^{10} x(t+2) + 12\bar{x}^2\right)$$

$$= \frac{1}{12}\left(\underbrace{\sum_{1}^{10} x(t)x(t+2)}_{27.82} - \bar{x}\left(\underbrace{\sum_{1}^{10}(x(t)+x(t+2))}_{33.2} - 12\bar{x}\right)\right)$$

$$= \frac{6.19}{12}$$

$$= 0.51583333\ldots$$

$\rho(\tau) = \frac{r(\tau)}{r(0)}$ för stationära processer

så $\rho(2)^* = \frac{r(2)^*}{r(0)^*} = \cancel{\frac{5.87}{3\cdot 12}\cdot\frac{12}{6.19}} = 0.316$ 18.57

602 a)

stationär pr. $\{X(n),\ n=0, \pm 1, \pm 2, \dots\}$

$$r(\tau) = \begin{cases} 1.2 & \tau = 0 \\ 0.5 & \tau = \pm 1 \\ 0 & \text{annars} \end{cases}$$

$$Y(n) = \frac{1}{N}\left(X(n) + X(n-1) + \dots + X(n-N+1)\right) = \frac{1}{N}\sum_{k=0}^{N-1} X(n-k)$$

$$Z(n) = \frac{1}{M}\left(X(n) + X(n-2) + \dots + X(n-2(M-1))\right) = \frac{1}{M}\sum_{k=0}^{M-1} X(n-2k)$$

Hur stort N för att $V(Y(n)) \le 0.1$?

$$V(Y(n)) = C\left(\frac{1}{N}\sum_{k=0}^{N-1} X(n-k),\ \frac{1}{N}\sum_{j=0}^{N-1} X(n-j)\right)$$

Tänk på $N \times N$-matris $C_{kj} = C(X(n-k), X(n-j))$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} C(X(n-k), X(n-j))$$

$$= \frac{1}{N^2}\Big(\underbrace{N r(0)}_{\text{diagonal-elementen}} + \underbrace{(N-1) r(-1)}_{\text{övre subdiagonalen}} + \underbrace{(N-1) r(1)}_{\text{undre subdiagonalen}}\Big)$$

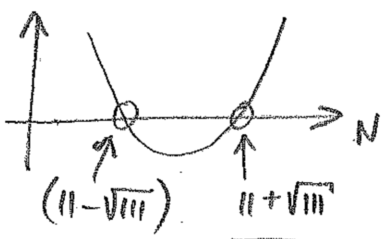
$$= \frac{1}{N^2}\left(1.2N + 2 \cdot 0.5(N-1)\right)$$

$$= \frac{2.2N - 1}{N^2} \le 0.1$$

$$2.2N - 1 \le 0.1 N^2$$

$$N^2 - \underbrace{\frac{2.2}{0.1}}_{22} N + \underbrace{\frac{1}{0.1}}_{10} \ge 0$$

$N = 11 \pm \sqrt{121 - 10}$




$N = 11 \pm \sqrt{111}$

Så välj $N \ge 11 + \sqrt{111}$ dvs $N \ge 22$

$$V(Z(n)) = \frac{1}{M^2} \sum_{k=0}^{M-1} \sum_{j=0}^{M-1} C(X(n-2k), X(n-2j))$$

$$= \frac{1}{M^2} \left( M r(0) + (M-2) \underbrace{r(-2)}_{=0} + (M-2) \underbrace{r(2)}_{=0} \right)$$

$$= \frac{1}{M} \cdot 1.2$$

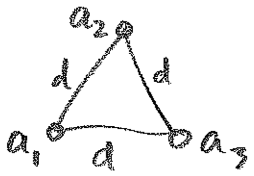
så för att $\frac{1.2}{M} \leq 0.1$ ska $M \geq \frac{1.2}{0.1} = 12$

b) Termerna i $Z(n)$ okorrelerade så behövs ej så stort $M$ (dvs färre termer) jämfört med $Y(n)$. Alltså $Z(n)$ "bättre" i nån mening.

604. Bullermätningar $\{Y_j\}$

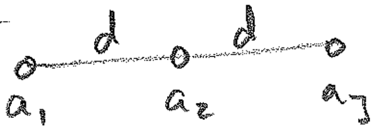
$$r_Y(d) = C(Y_i, Y_j) = e^{-d}, \quad d = |a_i - a_j|$$

Alt. a




$$C(Y_i, Y_j) = \begin{cases} 1 & \text{om } i = j \\ e^{-d} & \text{om } i \neq j \end{cases}$$

Alt. b




$$C(Y_i, Y_j) = \begin{cases} 1 & \text{om } i = j \\ e^{-d} & \text{om } |i-j| = 1 \\ e^{-2d} & \text{om } |i-j| = 2 \end{cases}$$

$$m^* = \frac{1}{3}(Y_1 + Y_2 + Y_3)$$

Alt. a (Kom ihåg: $m_1^*$ bättre än $m_2^*$ om $V(m_1^*) < V(m_2^*)$ (effektivare än))

$$V(m_a^*) = C\left(\frac{1}{3}(Y_1 + Y_2 + Y_3), \frac{1}{3}(Y_1 + Y_2 + Y_3)\right)$$

$$= \frac{1}{9}(3 \cdot 1 + 6 \cdot e^{-d}) = \frac{1 + 2e^{-d}}{3}$$

$$\Rightarrow 3V(m_a^*) - 1 - e^{-d} = e^{-d}$$

Alt. b

$$V(m_b^*) = \frac{1}{9}(3 \cdot 1 + 3e^{-d} + 3e^{-2d})$$

$$= \frac{1 + e^{-d} + e^{-2d}}{3} \Rightarrow 3V(m_b^*) - 1 - e^{-d} = e^{-2d}$$

Eftersom $e^{-2d} < e^{-d}$ då $d > 0$

$$3V(m_b^*) - 1 - e^{-d} = e^{-2d} < e^{-d} = 3V(m_a^*) - 1 - e^{-d}$$

Addera $1 + e^{-d}$ i alla led

och dividera med 3 så fås

$V(m_b^*) < V(m_a^*)$ dvs $m_b^*$ bättre än $m_a^*$.

606. svagt stationär $\{X(t)\}$

$m_X = m \qquad r(\tau) = e^{-\alpha|\tau|}, \quad \alpha > 0$

$m_1^* = \frac{1}{4}\int_0^4 X(t)\,dt \qquad m_2^* = \frac{1}{4}\sum_{t=1}^{4} X(t)$

a) $m_1^*, m_2^*$ vvr?

$$E(m_1^*) = \frac{1}{4} E\int_0^4 X(t)\,dt$$

$$= \frac{1}{4}\int_0^4 m\,dt$$

$$= \frac{1}{4}(4-0)\,m = m \quad \text{ok}$$

$$E(m_2^*) = \frac{1}{4}\sum_{t=1}^{4} E(X(t))$$

$$= \frac{1}{4}\,4m = m \quad \text{ok}$$

b) $$V(m_1^*) = \frac{1}{16} V\left(\int_0^4 X(t)\,dt\right)$$

$$= \frac{1}{16} C\left(\int_0^4 X(s)\,ds, \int_0^4 X(t)\,dt\right)$$

$$= \frac{1}{16}\int_0^4\int_0^4 r(s-t)\,ds\,dt$$

$$= \frac{1}{16}\int_0^4\int_0^4 e^{-\alpha|s-t|}\,ds\,dt$$

$$= \left\{\begin{matrix} u = s-t & s=0 \Leftrightarrow u=-t \\ du = ds & s=4 \Leftrightarrow u = 4-t \end{matrix}\right\}$$

$$= \frac{1}{16}\int_0^4\left(\int_{-t}^{4-t} e^{-\alpha|u|}\,du\right)dt$$

$$= \frac{1}{16}\int_0^4 \left( \int_{-t}^{0} e^{-\alpha(-u)}\,du + \int_0^{4-t} e^{-\alpha u}\,du \right) dt$$

$$= \frac{1}{16}\int_0^4 \left( \left[\frac{1}{\alpha} e^{\alpha u}\right]_{-t}^{0} + \left[-\frac{1}{\alpha} e^{-\alpha u}\right]_0^{4-t} \right) dt$$

$$= \frac{1}{16\alpha}\int_0^4 \left( \left(1 - e^{-\alpha t} - \left(e^{-\alpha(4-t)} - 1\right)\right)\right) dt$$

$$= \frac{1}{16\alpha}\int_0^4 \left(2 - e^{-\alpha t} - e^{\alpha t - 4\alpha}\right) dt$$

$$= \frac{1}{16\alpha}\left[2t + \frac{1}{\alpha} e^{-\alpha t} - \frac{1}{\alpha} e^{\alpha t - 4\alpha}\right]_0^4$$

$$= \frac{1}{16\alpha}\left(8 + \frac{1}{\alpha} e^{-4\alpha} - \frac{1}{\alpha} e^{4\alpha - 4\alpha} - 0 - \frac{1}{\alpha} + \frac{1}{\alpha}\right)$$

$$= \frac{8 + \frac{1}{\alpha} e^{-4\alpha} - \frac{1}{\alpha}}{16\alpha}$$

Fel på faktor 2

$$V(m_2^*) = V\left(\frac{1}{4}\sum_{t=1}^{4} X(t)\right)$$

$$= \frac{1}{16} C\left(\sum_1^4 X(s), \sum_1^4 X(t)\right)$$

$$= \frac{1}{16}\sum_1^4\sum_1^4 r(s-t)$$

$$= \frac{1}{16}\left(4r(0) + 6r(1) + 4r(2) + 2r(3)\right)$$

då $\begin{cases} s=1 \\ t=4 \end{cases}$ eller $\begin{cases} s=4 \\ t=1 \end{cases}$

$$= \frac{1}{16}\left(4\cdot 1 + 6e^{-\alpha} + 4e^{-2\alpha} + 2e^{-3\alpha}\right)$$

$$V(m_k^*) = \frac{1}{16}\int_0^4\int_0^4 e^{-\alpha|s-t|}\,ds\,dt$$

kolla först på $\displaystyle\int_0^4 e^{-\alpha|s-t|}\,ds =$

$$= \int_0^t e^{-\alpha|s-t|}\,ds + \int_t^4 e^{-\alpha|s-t|}\,ds$$

$$= \int_0^t e^{+\alpha(s-t)}\,ds + \int_t^4 e^{-\alpha(s-t)}\,ds$$

$$= e^{-\alpha t}\int_0^t e^{\alpha s}\,ds + e^{\alpha t}\int_t^4 e^{-\alpha s}\,ds$$

$$= e^{-\alpha t}\left[\frac{1}{\alpha}e^{\alpha s}\right]_0^t + e^{\alpha t}\left[-\frac{1}{\alpha}e^{-\alpha s}\right]_t^4$$

$$= e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha t}-1) - e^{\alpha t}\frac{1}{\alpha}(e^{-\alpha 4} - e^{-\alpha t})$$

$$= \frac{1}{\alpha}(1-e^{-\alpha t}) - \frac{1}{\alpha}(e^{\alpha t - 4\alpha} - 1)$$

$$= \frac{1}{\alpha}(1 - e^{-\alpha t} - e^{\alpha t - 4\alpha} + 1)$$

$$= \frac{1}{\alpha}(2 - e^{-\alpha t} - e^{\alpha t - 4\alpha})$$

$$16\alpha\left( V(m_1^*) - V(m_2^*)\right) =$$

$$= 8 + \frac{1}{\alpha}e^{-4\alpha} - \frac{1}{\alpha} - \left(4 + 6e^{-\alpha} + 4e^{-2\alpha} + 2e^{-3\alpha}\right)$$

$$= 4 - \frac{1}{\alpha} - 6e^{-\alpha} - 4e^{-2\alpha} - 2e^{-3\alpha} + \frac{1}{\alpha}e^{-4\alpha}$$

$\alpha = 1$: 0.17

$\alpha = 2$: 2.61

$r(\tau) = e^{-\alpha|\tau|}$

Varför kolla just dem?
Borde maximera m.a.p. $\alpha$
— numerisk analys!

Små $\alpha$ innebär starkare, tids-beroenden mer sammanhängande (pos.) kan räcka med $m_2^*$

Stora $\alpha$ innebär svagare beroenden och ev. större fluktuationer mellan heltalstidpunkterna så $m_1^*$ mer värdefull.

$\{X_t - m\}$ AR(1) process med ~~[illegible]~~ $\{e(t)\}$ standard normal vitt brus

607

a) $m^* = \bar{x}$

$= \frac{7.347}{10}$

$= 0.7347$

b) Efter som $\{e(t)\}$ normalfördelat är $\{X(t)\}$ en normalprocess och enl. ex 2 är $m^* \in N(m, \sqrt{V(m^*)})$ så ett 0.95 konf. int. för m är

$(m^* - \lambda_{0.05}\sqrt{V(m^*)}, \; m^* + \lambda_{0.05}\sqrt{V(m^*)})$

$V(m^*) = V\left(\frac{1}{10}\sum_1^{10} X(t)\right)$

$= \frac{1}{100} C\left(\sum X(t), \sum X(t)\right)$

$= \frac{1}{100} \sum_1^{10} \sum_1^{10} \underbrace{C(X(s), X(t))}_{r(s-t)}$

där $r(s-t)$ fås genom Y-W-ekv.

$\begin{cases} r(\tau) + 0.25\, r(\tau - 1) = 0 & t \geq 1 \\ r(0) + 0.25\, r(1) = 1 \end{cases}$ (ty ~~[illegible]~~ standard normal-vitt brus)

$r(0) = \frac{1}{1 - 0.25^2} = \frac{16}{15}$

$r(\tau) = (-0.25)^\tau \frac{16}{15} \quad \tau \geq 1$

så $V(m^*) = \frac{1}{100}\left(\frac{1}{15}\left(10 \cdot 16 - 18 \cdot 4 + 16 \cdot 1 - 14 \cdot \frac{1}{4} + \ldots + 4 \cdot \quad - 2 \cdot \frac{1}{16384}\right)\right)$