Solutions to Exam for Random Processes, 7.5 ECTS

April 28, 2011, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove that if $\{X_t\}$ is a weakly stationary random process in continuous time with covariance function $r_X(\tau)$, then the covariance function, $r_{X'}(\tau)$, of the derivative process, $\{X_t'\}$, is $-r_X''(\tau)$. (5p)

   Lösning: (See page 100 in the course literature.) □

2. Let $\{A_t\}$ be an $MA(2)$ process with $c_0 = c_2 = \sigma_\epsilon^2 = 1$ and $c_1 = 0$.
   (a) Determine the spectral density function $R_A(f)$ of $\{A_t\}$. (4p)
   (b) What kind of process is $\{B_t\}$ if $B_t = A_t + A_{t-1}$ for all t?
   Derive the covariance function $r_B(\tau)$ of $\{B_t\}$. (4p)

   Lösning:

   (a) Remembering that if $z = a + bi$, then $|z|^2 = a^2 + b^2$, we have that

$$
\begin{aligned}
R_A(f) &= 1^2 \cdot \left| \sum_{k=0}^{2} c_k e^{-i2\pi f k} \right|^2 \\
&= \left|1 \cdot e^{-i2\pi f \cdot 0} + 0 \cdot e^{-i2\pi f \cdot 1} + 1 \cdot e^{-i2\pi f \cdot 2}\right|^2 \\
&= \left|1 + e^{-i4\pi f}\right|^2 \\
&= |1 + \cos(-4\pi f) + i \sin(-4\pi f)|^2 \\
&= (1 + \cos(4\pi f))^2 + (-\sin(4\pi f))^2 \\
&= 1 + 2\cos(4\pi f) + \underbrace{\cos^2(4\pi f) + \sin^2(4\pi f)}_{=1} \\
&= 2(1 + \cos(4\pi f))
\end{aligned}
$$

   (b) Let $B_t = A_t + A_{t-1} = \epsilon_t + \epsilon_{t-2} + \epsilon_{t-1} + \epsilon_{t-3}$ for all t, so $\{B_t\}$ is an $MA(3)$ process with $c_0 = c_1 = c_2 = c_3 = \sigma_\epsilon^2 = 1$. Thus the cvf of $\{B_t\}$ is
$$
r_B(\tau) = \begin{cases} 4 & \text{if } \tau = 0 \\ 3 & \text{if } |\tau| = 1 \\ 2 & \text{if } |\tau| = 2 \\ 1 & \text{if } |\tau| = 3 \\ 0 & \text{otherwise} \end{cases}
$$
□

3. The *Gauss-Markov process* $\{Y_t : t \in \mathbb{R}^+\}$ is defined by $Y_t = e^{-\alpha t} W(e^{2\alpha t})$ for all $t \in \mathbb{R}^+$ where $\alpha > 0$ and $\{W(t)\}$ is a Wienerprocess with $V(W(1)) = \sigma^2$.

   (a) Prove that the process $\{Y_t\}$ is strongly stationary. (5p)

   (b) Let $\alpha = 0.1$ and $\sigma^2 = 1$ and calculate $V(\int_0^\infty Y_t\, dt)$. (4p)

Lösning:

(a) The covarinace function of a Wiener process $\{W(t)\}$ is $C(W(s), W(t)) = \sigma^2 \min(s,t)$. Thus for $\tau > 0$ we have that the cvf of $\{Y_t\}$ is

$$\begin{aligned}
r_Y(\tau) &= C(Y_t, Y_{t+\tau}) \\
&= C(e^{-\alpha t} W(e^{2\alpha t}),\ e^{-\alpha(t+\tau)} W(e^{2\alpha(t+\tau)})) \\
&= e^{-\alpha t} e^{-\alpha(t+\tau)} C(W(e^{2\alpha t}),\ W(e^{2\alpha(t+\tau)})) \\
&= e^{-\alpha t - \alpha t - \alpha \tau} \min(e^{2\alpha t},\ e^{2\alpha(t+\tau)}) \\
&= \sigma^2 e^{-2\alpha t} e^{-\alpha \tau} e^{2\alpha t} \\
&= \sigma^2 e^{-\alpha \tau}
\end{aligned}$$

Similarly when $\tau \leq 0$, $r_Y(\tau) = \sigma^2 e^{\alpha \tau}$. Thus for $\tau \in \mathbb{R}$ we have $r_Y(\tau) = \sigma^2 e^{-\alpha|\tau|}$. This is a function only of the time distance τ. Also $E(Y_t) = e^{-\alpha t} E(W(e^{2\alpha t})) = 0$. Therefore $\{Y_t\}$ is a weakly stationary process. Since it is a Gaussian process it is also strongly stationary.

(b) Let $X = \int_0^\infty Y_t\, dt$. Then, substituting x for $u - v$, we get

$$\begin{aligned}
V(X) &= C(X, X) \\
&= \int_0^\infty \int_0^\infty C(Y_u, Y_v)\, du\, dv \\
&= \int_0^\infty \int_0^\infty e^{-0.1|u-v|} du\, dv \\
&= \int_0^\infty \left(\int_{-v}^0 e^{-|x|} dx + \int_0^\infty e^{-|x|} dx \right) dv \\
&= \int_0^\infty \left(([-e^{-x}]_0^v + [-e^{-x}]_0^\infty)\, dv \right) \\
&= \int_0^\infty (2 - e^{-v})\, dv \\
&= \infty
\end{aligned}$$

□

4. Assume that $\{N_t : t \in \mathbb{R}^+\}$ is Poisson process with intensity $\lambda = 3$. Calculate

(a) $C(N_7, N_{11})$. (2p)

(b) $E(Y_t)$ where $Y_t \in Exp(N_t + 1)$. (6p)

Lösning:

(a) $C(N_7, N_{11}) = 3\min(7, 11) = 21$.

(b) $E(Y_t) = \int y\, f(y)\, dy$ where

$$\begin{aligned}
f(y) &= \frac{d}{dy} P(Y \le y) \\
&= \frac{d}{dy} \sum_{k=0}^{\infty} P(Y \le y \mid N_t = k) P(N_t = k) \\
&= \frac{d}{dy} \sum_{k=0}^{\infty} (1 - e^{-y(k+1)}) e^{-3} \frac{3^k}{k!} \\
&= e^{-3} \sum_{k=0}^{\infty} \frac{3^k}{k!} (k+1) e^{-y(k+1)} \\
&= e^{-3} \left(3e^{-2y} \sum_{k=0}^{\infty} \frac{(3e^{-y})^k}{k!} + e^{-y} \sum_{k=0}^{\infty} \frac{(3e^{-y})^k}{k!} \right) \\
&= e^{3(e^{-y}-1)-y} (3e^{-y} + 1)
\end{aligned}$$

Now, substituting t for e^{-y} the primitive of $f(y)$ is

$$\begin{aligned}
\int e^{3(e^{-y}-1)-y} (3e^{-y} + 1)\, dy &= \int e^{3(e^{-y}-1)-y} (3e^{-y} + 1)\, dy \\
&= -\int e^{3(t-1)} 3t\, dt - \int e^{3(t-1)} dt \\
&= -(t - \tfrac{1}{3}) e^{3(t-1)} - \tfrac{1}{3} e^{3(t-1)} \\
&= -e^{y+3(e^{-y}-1)}
\end{aligned}$$

Thus we finally get from integration by parts that

$$\begin{aligned}
E(Y_t) &= \int_0^{\infty} y\, e^{3(e^{-y}-1)-y} (3e^{-y} + 1)\, dy \\
&= \left[-ye^{-y+3(e^{-y}-1)} \right]_0^{\infty} + \int_0^{\infty} e^{-y+3(e^{-y}-1)} dy \\
&= \left[-ye^{-y+3(e^{-y}-1)} - \frac{1}{3} e^{3(e^{-y}-1)} \right]_0^{\infty} \\
&= \Big(-\lim_{y \to \infty} \underbrace{y\, e^{-y}}_{\to 0} e^{3(e^{-y}-1)} \Big) - \frac{1}{3} e^{3(0-1)} + 0 + \frac{1}{3} e^{3(1-1)} \\
&= \tfrac{1}{3}(1 - e^{-3})
\end{aligned}$$

□