

SOLUTIONS TO EXAM FOR STOCHASTIC MODELS IN DISCRETE TIME
3.75 ECTS

Master's program of Financial Mathematics
April 1, 2011, 9.00 – 13.00

Max number of points: 30.

Halmstad University grading bounds: 12p \Rightarrow grade 3, 18p \Rightarrow grade 4, 24p \Rightarrow grade 5.

ECTS bounds: 12p \Rightarrow grade E, 15p \Rightarrow grade D, 18p \Rightarrow grade C, 21p \Rightarrow grade B, 24p \Rightarrow grade A.

Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.

Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Let $X = \{(X_n, \mathcal{F}_n) : n \geq 0\}$ be a stochastic sequence with $E(|X_n|) < \infty$. Show that if X is a generalized martingale, then X is a martingale transformation. (5p)

Solution: (See pp 98–99 in *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(1)$ process with parameters $a_0 = 100$ and $a_1 = 0.9$. Determine the

(a) conditional probability $P(X_n \leq 111 | X_{n-1} = 11)$ if $\sigma_\epsilon^2 = 0.81$. (4p)

(b) noise variance σ_ϵ^2 such that the process $\{X_n\}$ is stationary. (5p)

Solution:

(a) Since $X_n = 100 + 0.9X_{n-1} + \sqrt{0.81}\epsilon_n$ for all n , we have that $P(X_n \leq 111 | X_{n-1} = 11) = P(100 + 0.9 \cdot 11 + 0.9\epsilon_n \leq 111) = P(\epsilon_n \leq \frac{111 - 100 - 0.9 \cdot 11}{0.9}) = \Phi(\frac{111 - 100 - 0.9 \cdot 11}{0.9})\Phi(1.22) = 0.8888$.

(b) To be stationary σ_ϵ^2 must be $1 - a_1^2 = 1 - 0.9^2 = 0.19$. □

3. Assume $\{h_n\}$ is distributed according to the $HARCH(p)$ model with $a_0 = 1$. Show that

$$\sum_{k=1}^p ka_k > 0 \tag{6p}$$

Solution: $h_n = \sigma_n \epsilon_n$ where $\sigma_n^2 = a_0 + \sum_{k=1}^p a_k \left(\sum_{j=1}^k h_{n-j}^2 + \sum_{i \neq j, i, j \leq k} h_{n-i} h_{n-j} \right)$. For $i < j$ we have $E(h_{n-i} h_{n-j}) = E(\epsilon_{n-i}) E(\sigma_{n-i} h_{n-j}) = 0$ so $E(h_n^2) = a_0 \sum_{k=1}^p a_k \sum_{j=1}^k E(h_{n-j}^2)$ but $E(h_n^2) = E(h_{n-j}^2) = c$ for all j so $c = a_0 + c \sum_{k=1}^p ka_k$ which implies that $E(h_n^2) = c = \frac{a_0}{1 - \sum_{k=1}^p ka_k} > 0$, i.e. $a_0 > 1 - \sum_{k=1}^p ka_k$. Now, if $a_0 = 1$ we have that $1 > 1 - \sum_{k=1}^p ka_k$, and thus $\sum_{k=1}^p ka_k > 0$. □

4. Consider the random walk $\{X_n : n = 0, 1, 2, \dots\}$.

(a) Calculate $V(X_3)$. (4p)

(b) Prove that the geometric random walk $\{Y_n : n = 0, 1, 2, \dots\}$, where $Y_n = e^{X_n}$, is a submartingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$. (6p)

Solution:

(a) $X_0 = 0, X_1 = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}, X_2 = \begin{cases} 2 & \text{w.p. } 1/4 \\ 0 & \text{w.p. } 1/2 \\ -2 & \text{w.p. } 1/4 \end{cases}, X_3 = \begin{cases} 3 & \text{w.p. } 1/8 \\ 1 & \text{w.p. } 3/8 \\ -1 & \text{w.p. } 3/8 \\ -3 & \text{w.p. } 1/8 \end{cases}.$

Therefore $E(X_3) = 0$ and $E(X_3^2) = 3^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + (-1)^2 \cdot \frac{3}{8} + (-3)^2 \cdot \frac{1}{8} = 3$ and thus $V(X_3) = E(X_3^2) - E(X_3)^2 = \frac{15}{4} - 0^2 = \frac{15}{4}$.

(b) Since $\{X_n\}$ is a random walk we can write $X_n = \sum_{k=1}^n \epsilon_k = X_{n-1} + \epsilon_n$ where $\{\epsilon_k\}$ are i.i.d. and $\epsilon_k = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$. Therefore $E(Y_n | \mathcal{F}_{n-1}) = E(e^{X_n} | X_1, X_2, \dots, X_{n-1}) = E(e^{X_{n-1} + \epsilon_n} | X_{n-1}) = e^{X_{n-1}} E(e^{\epsilon_n}) = e^{X_{n-1}} (e^{-1} \cdot \frac{1}{2} + e^{1} \cdot \frac{1}{2}) = e^{X_{n-1}} \cdot \frac{e^{-1} + e}{2} > e^{X_{n-1}} = Y_{n-1}$ because $e^{-1} + e > e > 2$. Thus $\{Y_n\}$ is a submartingale. □