①

$$y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = \frac{3}{4}x(n) - \frac{5}{8}x(n-1)$$

a) $\downarrow z$

$$Y(z) - \frac{9}{4}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) = \frac{3}{4}X(z) - \frac{5}{8}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{9}{4}z^{-1} + \frac{1}{2}z^{-2}\right] = X(z)\left[\frac{3}{4} - \frac{5}{8}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\frac{3}{4} - \frac{5}{8}z^{-1}\right) \times z^2}{\left(1 - \frac{9}{4}z^{-1} + \frac{1}{2}z^{-2}\right) \times z^2} = \frac{z\left(\frac{3}{4}z - \frac{5}{8}\right)}{\left(z^2 - \frac{9}{4}z + \frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{\frac{3}{4}z - \frac{5}{8}}{\left(z^2 - \frac{9}{4}z + \frac{1}{2}\right)} = \frac{\frac{3}{4}z - \frac{5}{8}}{(z-p_1)(z-p_2)} \overset{\text{pf.e.}}{=} \frac{A}{(z-p_1)} + \frac{B}{(z-p_2)}$$

Poles? $z^2 - \frac{9}{4}z + \frac{1}{2} = 0$

$$z_{1,2} = \frac{9}{8} \pm \sqrt{\left(\frac{9}{8}\right)^2 - \frac{1}{2}} = \frac{9}{8} \pm \frac{7}{8}$$

$$\begin{cases} p_1 = 2 \\ p_2 = \frac{1}{4} \end{cases}$$

zeros? $z\left(\frac{3}{4}z - \frac{5}{8}\right) = 0$

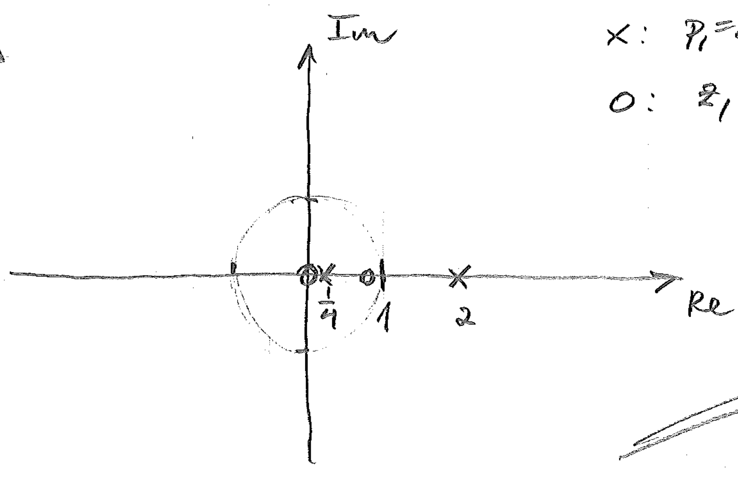
$\Rightarrow z_1 = 0$; $\frac{3}{4}z - \frac{5}{8} = 0 \Rightarrow z_2 = \frac{5}{6}$

$$A = (z-p_1)\frac{H(z)}{z}\Big|_{z=p_1} = \frac{\frac{3}{4}p_1 - \frac{5}{8}}{(p_1 - p_2)} = \frac{\frac{6}{4} - \frac{5}{8}}{\frac{7}{4}} = \frac{1}{2}$$

$$B = (z-p_2)\frac{H(z)}{z}\Big|_{z=p_2} = \frac{\frac{3}{4}\cdot p_2 - \frac{5}{8}}{(p_2 - p_1)} = \frac{\frac{3}{16} - \frac{5}{8}}{-\frac{7}{4}} = \frac{1}{4}$$

So, $$H(z) = \frac{\frac{1}{2}}{(1 - 2\cdot z^{-1})} + \frac{\frac{1}{4}}{(1 - \frac{1}{4}z^{-1})}$$

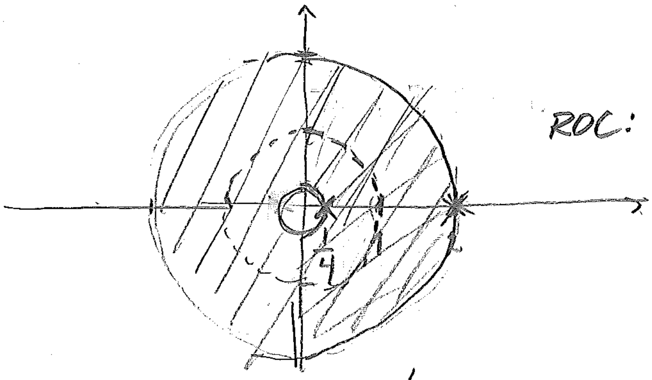
Pole-zero pattern:



×: $p_1 = 2$; $p_2 = \frac{1}{4}$ poles

o: $z_1 = 0$, $z_2 = \frac{5}{6}$ zeros

b) The system is ~~stable~~ $\Rightarrow$ $|z| = 1$ is included in the ROC



ROC: $\frac{1}{4} < |z| < 2$

No poles in the ROC

$$H(z) = \frac{\frac{1}{2}}{(1-2z^{-1})} + \frac{\frac{1}{4}}{(1-\frac{1}{4}z^{-1})}$$

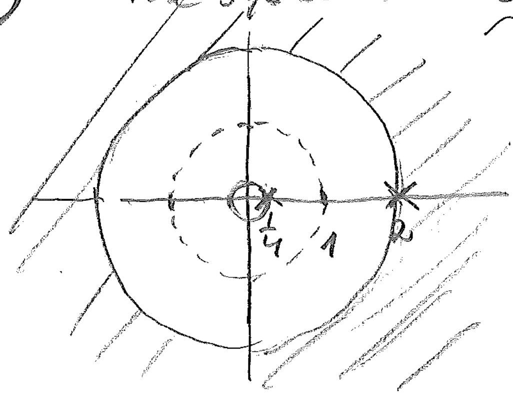
anticausal causal

z^{-1}

$$h(n) = -\frac{1}{2}(2)^n u(-n-1) + \frac{1}{4}\left(\frac{1}{4}\right)^n \cdot u(n)$$

two-sided signal

c) The system is causal $\Rightarrow$ $|z| > r$



ROC: $|z| > 2$

No poles in the ROC

$$H(z) = \frac{\frac{1}{2}}{(1-2z^{-1})} + \frac{\frac{1}{4}}{(1-\frac{1}{4}z^{-1})}$$

z^{-1}

$$h(n) = \frac{1}{2}(2)^n \cdot u(n) + \frac{1}{4}\left(\frac{1}{4}\right)^n \cdot u(n)$$

causal signal.

(2) $h(n) = \{\underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3}\}$

a) $$H(\omega) = \sum_{n=0}^{2} h(n)\, e^{-j\omega n}$$

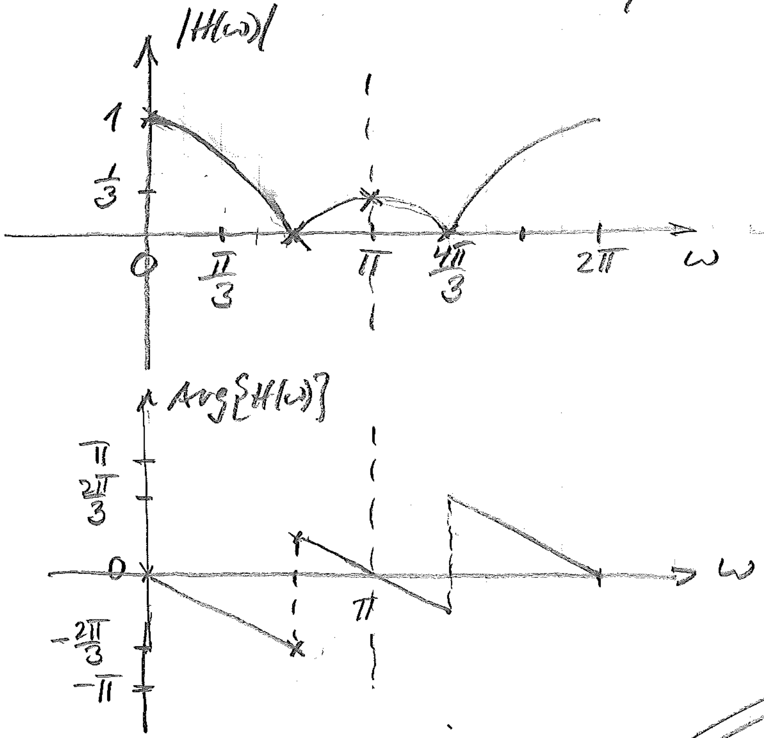
$$= \tfrac{1}{3} + \tfrac{1}{3} e^{-j\omega} + \tfrac{1}{3} e^{-j\omega 2}$$

$$= \tfrac{1}{3} e^{-j\omega} [\underbrace{e^{j\omega} + e^{-j\omega}}_{2\cdot\cos(\omega)}] + \tfrac{1}{3} e^{-j\omega}$$

$$= e^{-j\omega} \cdot \underbrace{\tfrac{1}{3}[1 + 2\cos(\omega)]}_{H_{real}(\omega)}$$

$$|H(\omega)| = \tfrac{1}{3}\,|1 + 2\cdot\cos(\omega)|$$

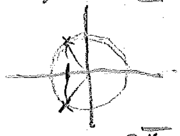
$$\mathrm{Arg}\{H(\omega)\} = -\omega + \underbrace{\mathrm{Arg}\{1 + 2\cos(\omega)\}}_{= 0/\pi}$$



Zero-crossing?

$2\cos(\omega) = -1$

$\cos(\omega) = -\tfrac{1}{2}$



$\omega = \pm \tfrac{2\pi}{3}$

$\omega = 0 \Rightarrow |H(0)| = 1$

$\omega = \pi \Rightarrow |H(\pi)| = 1/3$

② cont.

b) $$H(\omega) = e^{-j\omega}\,\frac{1}{3}\{1 + 2\cdot\cos(\omega)\}$$

Steady state response to sine/cosine inputs:

$$y(n) = |H(\tfrac{\pi}{3})|\cdot 2\sin\left(\tfrac{\pi}{3}(n-1) + \mathrm{Arg}\{H(\tfrac{\pi}{3})\}\right) + |H(\tfrac{2\pi}{3})|\cdot 0.5\cos\left(\tfrac{2\pi}{3}\cdot n + \mathrm{Arg}\{H(\tfrac{2\pi}{3})\}\right)$$

$$\begin{cases} H(\tfrac{2\pi}{3}) = e^{-j\frac{2\pi}{3}}\,\frac{1}{3}\{1 + 2\cdot\underbrace{\cos(\tfrac{2\pi}{3})}_{=-\frac{1}{2}}\} = 0 & ;\ \text{see } |H(\omega)| \\ H(\tfrac{\pi}{3}) = e^{-j\frac{\pi}{3}}\,\frac{1}{3}[1 + 2\cdot\underbrace{\cos(\tfrac{\pi}{3})}_{=\frac{1}{2}}] & \\ \quad = e^{-j\frac{\pi}{3}}\cdot\frac{2}{3} & ;\ \text{see } |H(\omega)| \text{ and } \mathrm{Arg}\{H(\omega)\} \end{cases}$$

$$y(n) = \frac{2}{3}\cdot 2\sin\left(\frac{\pi}{3}(n-1) - \frac{\pi}{3}\right)$$

$$= \frac{4}{3}\sin\left(\frac{\pi}{3}(n-2)\right)$$

③

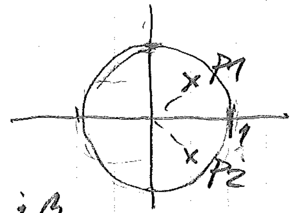
a) $$H(z) = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} \begin{matrix} \times z^2 \\ \times z^2 \end{matrix} = \frac{z^2}{z^2 - z + \frac{1}{2}}$$

$$\frac{H(z)}{z} = \frac{z}{(z-p_1)(z-p_2)} \overset{\text{p.f.e.}}{=} \frac{A}{(z-p_1)} + \frac{A^*}{(z-p_1^*)}$$

Poles p_1 and p_2 ?

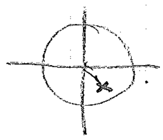
$$z^2 - z + \frac{1}{2} = 0$$

$$z_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{2}} = \frac{1}{2} \pm j\frac{1}{2}$$



$$\begin{cases} p_1 = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} = r_1 \cdot e^{j\beta} \\ p_2 = p_1^* \end{cases}$$

$$A = (z - p_1)\frac{H(z)}{z}\Big|_{z=p_1} = \frac{p_1}{p_1 - p_1^*} = \frac{\frac{1}{2} + j\frac{1}{2}}{j} = \frac{1}{2} - j\frac{1}{2} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$$



$$H(z) = \frac{A}{(1 - p_1 z^{-1})} + \frac{A^*}{(1 - p_1^* z^{-1})}$$

$\mathcal{Z}^{-1}$

$$h(n) = 2 \cdot |A| \cdot (r_1)^n \cos(\beta \cdot n + \mathrm{Arg}\{A\}) \cdot u(n)$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4} \cdot n - \frac{\pi}{4}\right) \cdot u(n)$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^{n+1} \cos\left(\frac{\pi}{4}(n-1)\right) u(n)$$

---

b) Compute $y(n)$ when $x(n) = u(n)$

$$Y(z) = H(z) \cdot X(z) \quad ; \quad X(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{(1 - z^{-1} + \frac{1}{2}z^{-2})} \cdot \frac{1}{1 - z^{-1}} \quad \begin{matrix} \times z^3 \\ \times z^3 \end{matrix}$$

$$= \frac{z^3}{(z - p_1)(z - p_1^*)(z - 1)}$$

③ cont.

$$\frac{Y(z)}{z} = \frac{z^2}{(z-p_1)(z-p_1^*)(z-1)} \overset{\text{pfe}}{=} \frac{A_1}{(z-p_1)} + \frac{A_1^*}{(z-p_1^*)} + \frac{B}{(z-1)}$$

$$A_1 = (z-p_1)\cdot\frac{Y(z)}{z}\Big|_{z=p_1} = \frac{p_1^2}{(p_1-p_1^*)(p_1-1)}$$

$$= \frac{\left(\frac{1}{2}+j\frac{1}{2}\right)\left(\frac{1}{2}+j\frac{1}{2}\right)}{j\left(-\frac{1}{2}+j\frac{1}{2}\right)} = \frac{j\frac{1}{2}}{j\left(-\frac{1}{2}+j\frac{1}{2}\right)} = \frac{\frac{1}{2}}{-\frac{1}{2}(1-j)}$$

$$= \frac{-(1+j)}{(1-j)(1+j)} = \frac{-(1+j)}{2} = -\frac{1}{2}-j\frac{1}{2}$$

$$= \frac{1}{\sqrt{2}}e^{-j\frac{3\pi}{4}}$$

$$B = (z-1)\frac{Y(z)}{z}\Big|_{z=1} = \frac{1^2}{(1-p_1)(1-p_1^*)} =$$

$$= \frac{1}{(1\underbrace{-p_1^*-p_1}+p_1p_1^*)} = \frac{1}{1-1+\frac{1}{2}} = 2$$

$$(p_1^*+p_1) = 1$$

$$p_1p_1^* = |p_1|^2 = \frac{1}{2}$$

$$Y(z) = \frac{A_1}{(1-p_1z^{-1})} + \frac{A_1^*}{(1-p_1^*z^{-1})} + \frac{2}{(1-z^{-1})}$$

$\mathcal{Z}^{-1}$

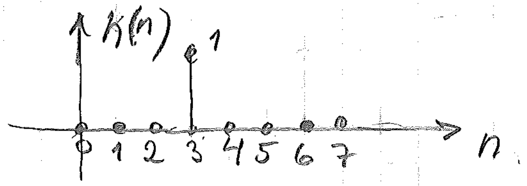
$$y(n) = 2\cdot|A_1|(r_1)^n\cos(\theta\cdot n + \operatorname{Arg}\{A_1\})\cdot u(n) + 2\cdot u(n)$$

$$= 2\cdot\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n-\frac{3\pi}{4}\right)\cdot u(n) + 2\cdot u(n)$$

$$= 2\cdot\left(\frac{1}{\sqrt{2}}\right)^{n+1}\cos\left(\frac{\pi}{4}(n-3)\right)\cdot u(n) + 2\cdot u(n)$$

(4)

$h(n) = \delta(n-3)$; causal system



a) Compute the 8 point DFT $\Rightarrow N=8$

$$H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n} \qquad k = 0,1,2,\dots N-1$$

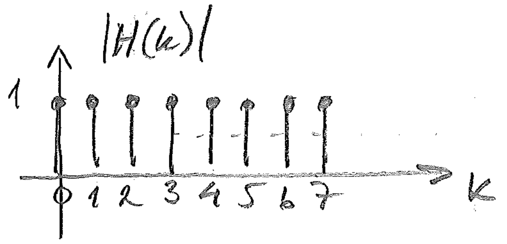
$$= \sum_{n=0}^{7} \delta(n-3)\, e^{-j\frac{2\pi}{8}\cdot k\cdot n} = 1\cdot e^{-j\frac{2\pi}{8}\cdot k\cdot 3}$$

$$= 1\, e^{-j\frac{3\pi}{4}\cdot k} \quad ; k=0,1,2,\dots,7$$

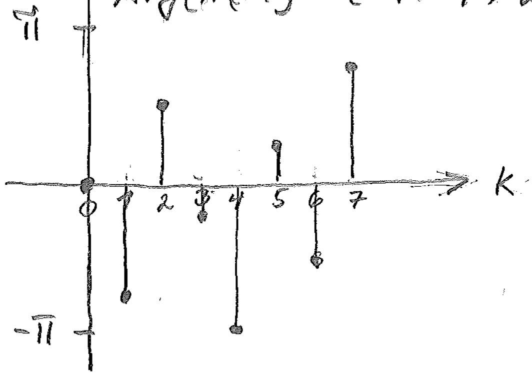
b) Magnitude $|H(k)| = 1$

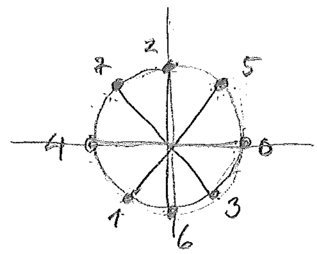
Phase $\mathrm{Arg}\{H(k)\} = -\frac{3\pi}{4}\cdot k$

$k = 0,1,2,\dots,7$



$\mathrm{Arg}\{H(k)\} = \{0, -\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{4}, -\pi, \frac{\pi}{4}, -\frac{\pi}{2}, \frac{3\pi}{4}\}$





c)

$x(n) \rightarrow \boxed{\delta(n-3)} \rightarrow y(n)$ $\quad y(n) = x(n) * \delta(n-3) = x(n-3)$ pure delay!

$x(n) = [0.8\cos(\frac{3\pi}{4}\cdot n) + 1.2]\, u(n)$

$\Rightarrow y(n) = [0.8\cos(\frac{3\pi}{4}(n-3)) + 1.2]\, u(n-3)$