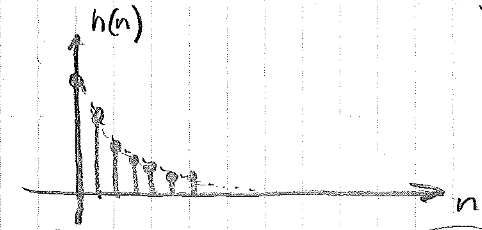
(1)

$$h(n) = a(b)^n \cdot u(n) \quad ; \; b>0$$

a) BIBO-stable system: $\sum_{n=0}^{\infty} |h(n)| < \infty$

$$\sum_{n=0}^{\infty} |a|(b)^n = |a| \sum_{n=0}^{\infty} (b)^n = |a| \cdot \frac{1}{1-b}$$

if $|b| < 1$



$\xRightarrow{b>0}$ $\underline{\underline{b<1}}$

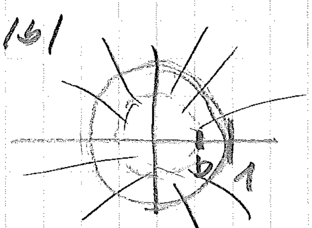
b)

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a(b)^n z^{-n} = a \sum_{n=0}^{\infty} (b \cdot z^{-1})^n = a \frac{1}{1-bz^{-1}}$$

if $\left|\frac{b}{z}\right| < 1$

$\Longleftrightarrow |z| > |b|$



$|z|=1 \in$ ROC of $H(z)$ if $|b|<1$.

$$\boxed{H(\omega) = H(z)\Big|_{z=e^{j\omega}}} = \frac{a}{(1-b\cdot e^{-j\omega})}$$

$$H(0) = \frac{a}{(1-b)} \Rightarrow \underline{\underline{a = 1-b}}$$

$H(0)=1$

c)

$$H(z) = \frac{a}{(1-bz^{-1})}$$

$$\frac{Y(z)}{X(z)} = \frac{a}{(1-bz^{-1})}$$

$$Y(z)[1-bz^{-1}] = a \cdot X(z)$$

$$Y(z) - bz^{-1}Y(z) = a \cdot X(z)$$

z^{-1}

$$\boxed{y(n) - by(n-1) = a \cdot x(n)}$$

②

$$\xrightarrow{z^+}\quad \boxed{y(n)} + y(n-1) + 0.5\,y(n-2) = \boxed{x(n)} \quad ; \quad y(-1) = 2,\ \ y(-2) = 1$$

$$Y(z) + z^{-1}Y(z) + y(-1) + 0.5\left(z^{-2}Y(z) + y(-2) + y(-1)\cdot z^{-1}\right) = X(z)$$

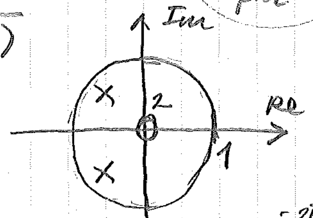
$$Y(z)\left(1 + z^{-1} + 0.5z^{-2}\right) = X(z) - y(-1) - 0.5\,y(-2) - 0.5\,z^{-1}y(-1)$$

$$Y(z) = \underbrace{\frac{1}{(1+z^{-1}+0.5z^{-2})}}_{H(z)} \cdot X(z) + \frac{-y(-1) - 0.5\,y(-2) - 0.5\,z^{-1}y(-1)}{(1+z^{-1}+0.5z^{-2})}$$

a)

$$H(z) = \frac{1}{(1+z^{-1}+0.5z^{-2})}\ \frac{\times z^2}{\times z^2} = \frac{z^2}{(z^2+z+0.5)}$$

Pole-zero plot



× zeros: $z^2 = 0 \longrightarrow z_1 = z_2 = 0$

× poles: $z^2 + z + 0.5 = 0$

$$p_{1,2} = -\frac{1}{2} \pm \underbrace{\sqrt{\frac{1}{4} - \frac{1}{2}}}_{j\frac{1}{2}} \Rightarrow \begin{cases} p_1 = -\frac{1}{2} + j\frac{1}{2} = \frac{1}{\sqrt{2}}e^{j\frac{3\pi}{4}} \\ p_2 = -\frac{1}{2} - j\frac{1}{2} \end{cases}$$

b)

$$\frac{H(z)}{z} = \frac{z}{(z^2+z+0.5)} = \frac{z}{(z-p_1)(z-\overline{p}_1)} = \frac{c_1}{(z-p_1)} + \frac{\overline{c}_1}{(z-\overline{p}_1)}$$

$$\begin{cases} c_1 = (z-p_1)\frac{H(z)}{z}\Big|_{z=p_1} = \frac{p_1}{(p_1-\overline{p}_1)} = \frac{-\frac{1}{2}+j\frac{1}{2}}{j}\ \frac{\cdot j}{\cdot j} \\ \qquad = \frac{-\frac{1}{2}-j\frac{1}{2}}{-1} = \frac{1}{2}+j\frac{1}{2} = \frac{1}{\sqrt{2}}e^{j\pi/4} \\ \overline{c}_1 = \frac{1}{2} - j\frac{1}{2} \end{cases}$$

$$H(z) = \frac{c_1}{(1-p_1z^{-1})} + \frac{\overline{c}_1}{(1-\overline{p}_1z^{-1})}$$

$$\xrightarrow{z^{-1}}\quad p_1 = r\cdot e^{j\omega} = \frac{1}{\sqrt{2}}e^{j3\pi/4}$$

$$c_1 = |c_1|e^{j\angle c_1} = \frac{1}{\sqrt{2}}\cdot e^{j\pi/4}$$

$$h(n) = 2\cdot|c_1|\cdot(r)^n\cos(\omega\cdot n + \angle c_1)\cdot u(n)$$

$$= 2\cdot\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{3\pi}{4}\cdot n + \frac{\pi}{4}\right)\cdot u(n)$$

② forts.

c)

$$Y(z) = \frac{1}{(1+z^{-1}+0{,}5z^{-2})}\cdot X(z) + \frac{-2-0{,}5\cdot 1-0{,}5z^{-1}\cdot 2}{(1+z^{-1}+0{,}5z^{-2})}$$

$$= \underbrace{\frac{1}{(1+z^{-1}+0{,}5z^{-2})}\cdot\frac{1}{(1-z^{-1})}}_{Y_1(z)} + \underbrace{\frac{-2{,}5-z^{-1}}{(1+z^{-1}+0{,}5z^{-2})}}_{Y_2(z)}$$

$$Y_1(z) = \frac{z^3}{(z^2+z+0{,}5)(z-1)}$$

$$\frac{Y_1(z)}{z} = \frac{z^2}{(z-p_1)(z-\bar{p}_1)(z-1)} = \frac{c_1}{(z-p_1)} + \frac{\bar{c}_1}{(z-\bar{p}_1)} + \frac{c_3}{(z-1)}$$

$$c_1 = (z-p_1)\frac{Y_1(z)}{z}\bigg|_{z=p_1} = \frac{p_1^2}{(p_1-\bar{p}_1)(p_1-1)} = \frac{\left(-\frac{1}{2}+j\frac{1}{2}\right)\left(-\frac{1}{2}+j\frac{1}{2}\right)}{j\left(-\frac{3}{2}+j\frac{1}{2}\right)}$$

$$= \frac{-j\frac{1}{2}}{j\left(-\frac{3}{2}+j\frac{1}{2}\right)} = \frac{-\frac{1}{2}\left(-\frac{3}{2}-j\frac{1}{2}\right)}{5/2} = \underline{\frac{3}{10}+j\frac{1}{10}}$$

$$\bar{c}_1 = \frac{3}{10} - j\frac{1}{10}$$

$$c_3 = (z-1)\frac{Y_1(z)}{z}\bigg|_{z=1} = \frac{1^2}{(1-p_1)(1-\bar{p}_1)} = \frac{1}{\left(\frac{3}{2}-j\frac{1}{2}\right)\left(\frac{3}{2}+j\frac{1}{2}\right)}$$

$$= \frac{1}{\frac{9}{4}+\frac{1}{4}} = \underline{\frac{4}{10}}$$

$$Y_1(z) = \frac{c_1}{(1-p_1z^{-1})} + \frac{\bar{c}_1}{(1-\bar{p}_1z^{-1})} + \frac{c_3}{(1-z^{-1})}$$

z^{-1}

$$p_1 = r\cdot e^{j\omega} = \frac{1}{\sqrt{2}}e^{j3\pi/4}$$

$$c_1 = |c_1|e^{j\angle c_1} = 0{,}3162\cdot e^{j0{,}3218}$$

$$y_1(n) = \left(2\cdot|c_1|(r)^n\cos(\omega\cdot n+\angle c_1) + c_3\right)\cdot u(n)$$

$$= \left(2\cdot 0{,}3162\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{3\pi}{4}\cdot n + 0{,}3218\right) + 0{,}4\right)\cdot u(n)$$

② c)

forts. $Y_2(z) = \dfrac{-2{,}5 - z^{-1}}{(1+z^{-1}+0{,}5z^{-2})} \begin{matrix} \times z^2 \\ \times z^2 \end{matrix} \quad \dfrac{-2{,}5z^2 - z}{(z^2+z+0{,}5)}$

$$\frac{Y_2(z)}{z} = \frac{-2{,}5z - 1}{(z-p_1)(z-\bar{p}_1)} = \frac{c_1}{(z-p_1)} + \frac{\bar{c}_1}{(z-\bar{p}_1)}$$

$$\begin{cases} c_1 = (z-p_1)\dfrac{Y_2(z)}{z}\Big|_{z=p_1} = \dfrac{-2{,}5\left(-\frac{1}{2}+j\frac{1}{2}\right) - 1}{(p_1 - \bar{p}_1)} \\ \quad = \dfrac{\frac{1}{4} - j\frac{5}{4}}{j}\cdot\dfrac{\cdot j}{\cdot j} = \dfrac{\frac{5}{4} + j\frac{1}{4}}{-1} = -\frac{5}{4} - j\frac{1}{4} \\ \bar{c}_1 = -\frac{5}{4} + j\frac{1}{4} \end{cases}$$

$$Y_2(z) = \frac{c_1}{(1-p_1z^{-1})} + \frac{\bar{c}_1}{(1-\bar{p}_1z^{-1})}$$

z^{-1}

$$p_1 = re^{j\omega} = \frac{1}{\sqrt{2}}e^{j3\pi/4}$$

$$c_1 = |c_1|e^{j\angle c_1} = 1{,}2748\,e^{-j2{,}9442}$$

$$y_2(n) = 2\cdot|c_1|\cdot(r)^n\cos(\omega\cdot n + \angle c_1)\cdot u(n)$$

$$= 2\cdot 1{,}2748\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{3\pi}{4}\cdot n - 2{,}9442\right)u(n)$$

$$y(n) = y_1(n) + y_2(n)$$

$$= \left(2\cdot 0{,}3162\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{3\pi}{4}\cdot n + 0{,}3218\right) + 0.4\right)u(n)$$

$$+ 2\cdot 1{,}2748\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{3\pi}{4}\cdot n - 2.9442\right)\cdot u(n)$$

③ $h(n) = \{ \underset{\uparrow}{\tfrac{1}{4}}, \tfrac{1}{2}, \tfrac{1}{4} \}$

a) $H(\omega) = \sum_{n=0}^{2} h(n) e^{-j\omega n}$

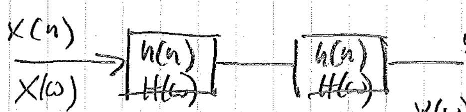
$= \frac{1}{4} + \frac{1}{2} e^{-j\omega} + \frac{1}{4} e^{-j\omega 2} = \frac{1}{4} e^{-j\omega} \underbrace{\left(e^{j\omega} + e^{-j\omega}\right)}_{2\cdot\cos(\omega)} + \frac{1}{2} e^{-j\omega}$

$= e^{-j\omega} \cdot \frac{1}{2}\left(1 + \cos(\omega)\right)$

$|H(\tfrac{5\pi}{16})| = \frac{1}{2}\left(1 + \cos(\tfrac{5\pi}{16})\right) = \underline{\underline{0{,}7778}}$

---

b) $h2(n) = h(n) * h(n)$



$y(n) = x(n) * \underbrace{h(n) * h(n)}_{h_2(n)}$

$Y(\omega) = X(\omega) \cdot \underbrace{H(\omega) \cdot H(\omega)}_{H^2(\omega)}$

$\mathcal{Z}$

$H2(z) = H(z) \cdot H(z)$

$\boxed{H(\omega) = H(z)\big|_{z=e^{j\omega}}} \rightarrow H2(\omega) = H(\omega) \cdot H(\omega) = \underline{\underline{H^2(\omega)}}$

$H2(\omega) = |H(\omega)| e^{j\angle H(\omega)} \cdot |H(\omega)| \cdot e^{j\angle H(\omega)}$

$= |H(\omega)|^2 e^{j2\cdot\angle H(\omega)}$

$h2(n) = \{ \underset{\uparrow}{\tfrac{1}{16}}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{4}, \tfrac{1}{16} \}$

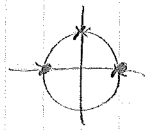
$|H2(\omega)| = |H(\omega)|^2 = |H(\tfrac{5\pi}{16})|^2 = \underline{\underline{0{,}6049}}$

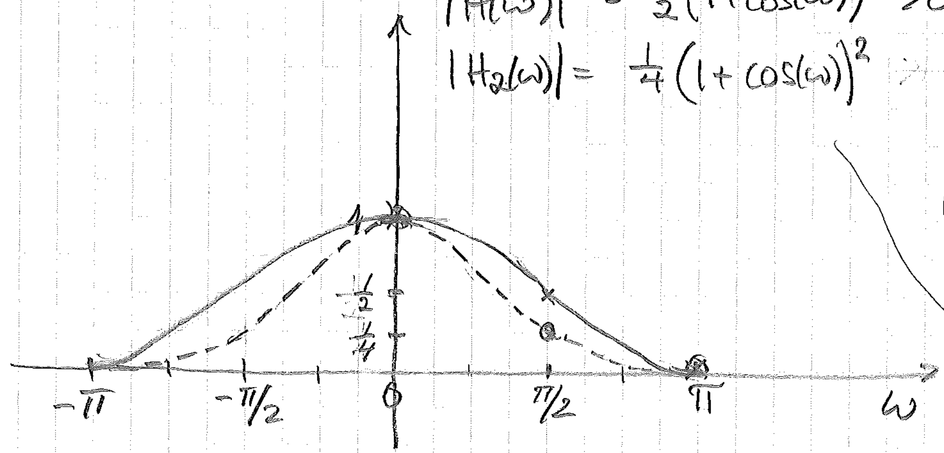
---

c)

$|H(\omega)| = \frac{1}{2}(1 + \cos(\omega)) \geq 0$ for all ω!

$|H_2(\omega)| = \frac{1}{4}(1 + \cos(\omega))^2$





$|H(0)| = 1$; $|H_2(0)| = 1$

$|H(\pi/2)| = \frac{1}{2}$; $|H_2(\pi/2)| = \frac{1}{4}$

$|H(\pi)| = 0$; $|H_2(\pi)| = 0$

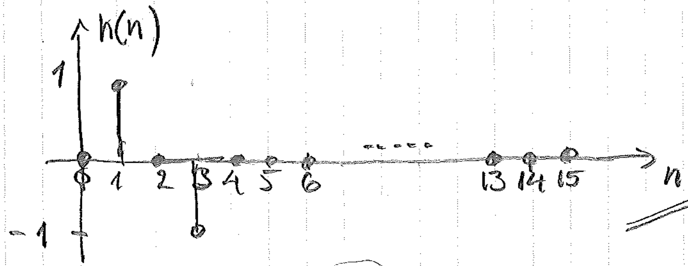
+ cosine

---

④

$$h(n) = \big(\delta(n-1) - \delta(n-3)\big)\cdot\big(u(n) - u(n-10)\big) \quad ; \quad \underline{N=16}$$

a)



$$H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n} = \sum_{n=0}^{15} h(n)\, e^{-j\frac{2\pi}{16}\cdot k\cdot n}$$

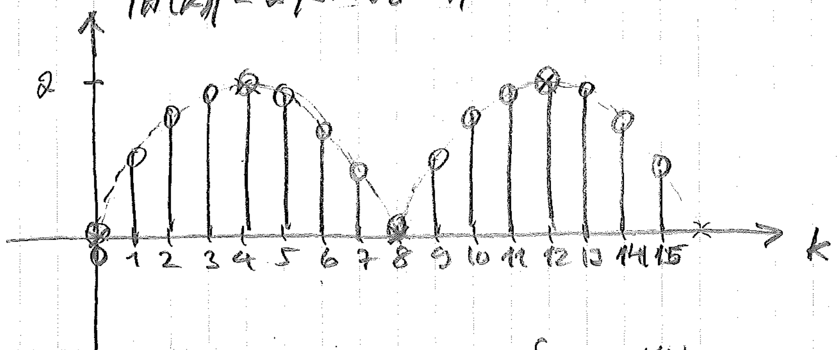
$$= \underbrace{e^{-j\frac{\pi}{8}\cdot k\cdot 1}}_{n=1} - \underbrace{e^{-j\frac{\pi}{8}\cdot k\cdot 3}}_{n=3}$$

$$= e^{-j\frac{\pi}{8}\cdot k\cdot 2}\underbrace{\left(e^{j\frac{\pi}{8}k} - e^{-j\frac{\pi}{8}\cdot k}\right)}_{j\cdot 2\sin\left(\frac{\pi}{8}\cdot k\right)}$$

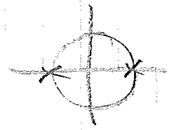
$$= e^{-j\left(\frac{\pi}{4}k - \frac{\pi}{2}\right)}\underbrace{2\cdot\sin\left(\frac{\pi}{8}\cdot k\right)}_{H(k)_{real}}$$

b)

$$|H(k)| = 2\left|\sin\left(\tfrac{\pi}{8}\cdot k\right)\right|$$



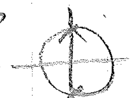
× zeros?



$\frac{\pi}{8}\cdot k = m\cdot\pi$ $\quad m=0,1,2,3$

$k = m\cdot 8$

× Max = 2 ?

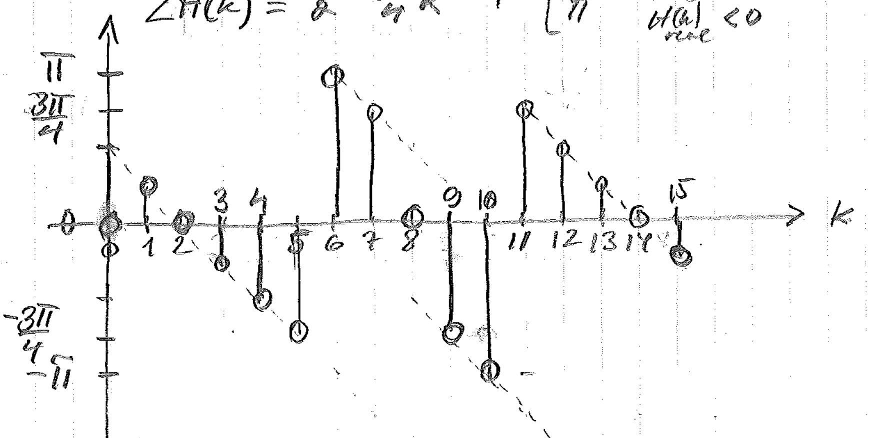


$\frac{\pi}{8}\cdot k = m\cdot\frac{\pi}{2}$ $\quad m=1,3,5,7$

$k = m\cdot 4$

× sine

$$\angle H(k) = \frac{\pi}{2} - \frac{\pi}{4}k + \begin{cases} 0 & H(k)_{real} > 0 \\ \pi & H(k)_{real} < 0 \end{cases}$$



× $H(0) = 0 \Rightarrow \angle H(0) = 0$

× $H(8) = 0 \Rightarrow \angle H(8) = 0$

× $H(2) = \text{real} > 0 \Rightarrow \angle H(2) = 0$

+ kompl. conj.

+ symmetry around $k = \frac{N}{2} = 8$

c)

$$x(n) = 1.5\cos\left(\tfrac{5\pi}{8}n\right) \qquad \omega = \frac{5\pi}{8} \qquad \omega_k = \frac{2\pi}{N}\cdot k = \frac{2\pi}{16}\cdot k = \frac{5\pi}{8}$$

$\underline{k=5}$

$$y_{ss}(n) = \underbrace{|H(5)|}_{1.8478}\cdot 1.5\cos\left(\tfrac{5\pi}{8}n + \underbrace{\angle H(5)}_{-\frac{3\pi}{4}}\right)$$