Solutions to Exam for Random Processes, 7.5 ECTS

January 15, 2011, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume that $\{X_t\}$ is a weakly stationary process which is filtered through a linear filter with frequency function H resulting in the output signal $\{Y_t\}$. Prove that $E(Y_t) = H(0)E(X_t)$. (3p)

   Solution: By the definition of linear filtering with impulse response $Y_t = = \int h(u)X_{t-u}du$ where $H(f) = \int h(u)e^{-i2\pi fu}du$. Since $\{X_t\}$ is weakly stationary $E(X_t)$ is constant w.r.t. t and thus $E(Y_t) = E(\int h(u)X_{t-u}du) = \int h(u)E(X_{t-u})\,du = E(X_t)\int h(u)\,du = E(X_t)\int h(u)e^{-i2\pi\cdot 0\cdot u}du = E(X_t)H(0)$. □

2. Let $\{X_t : t \in \mathbb{R}\}$ be a weakly stationary process with spectral density function

$$R(f) = \begin{cases} 2-|f| & \text{if } |f| \le 2 \\ 0 & \text{if } |f| > 2 \end{cases}$$

   Calculate

   (a) $V(X_t)$. (2p)

   (b) the cvf of $\{X_t\}$: $r_X(\tau)$. (4p)

   Now assume that $\{X_t\}$ is sampled at timepoints $t = \ldots, -4, -2, 0, 2, 4, \ldots$

   (c) Determine the spectral density of the sampled signal. (4p)

   Solution:

   (a) Since $r(\tau) = \int e^{i2\pi f\tau}R(f)\,df$ we have that $V(X_t) = r(0) = \int R(f)\,df = = \int_{-2}^{2}(2-|f|)\,df = 2\int_0^2(2-f)\,df = 2[2f - f^2/2]_0^2 = 2(4-4/2-0) = 4$.

   (b) $R_X(f) = 2(1-\frac{1}{2}|f|)I(|f| \le 2)$. Then, according to the tables, we have that $\mathcal{F}((1-\frac{1}{2}|f|)I(|f| \le 2)) = \begin{cases} 2 & \text{if } f = 0 \\ \frac{1-\cos(4\pi f)}{2\pi^2 f^2} & \text{if } f \neq 0 \end{cases}$ and thus, since the covariance function is even, $r_X(\tau) = 2\begin{cases} 2 & \text{if } \tau = 0 \\ \frac{1-\cos(4\pi\tau)}{4\pi^2\tau^2} & \text{if } \tau \neq 0 \end{cases} = \begin{cases} 4 & \text{if } \tau = 0 \\ \frac{1-\cos(4\pi\tau)}{2\pi^2\tau^2} & \text{if } \tau \neq 0. \end{cases}$

   (c) $Z_t = X_t$ for $t = 0, \pm 2, \pm 4, \ldots$ i.e. $d = 2$ in the notation of the book and the summary of formulae. Thus $R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f+\frac{k}{2})$ for $-\frac{1}{4} < f \le \frac{1}{4}$. Now $R_X(f)$ is non-zero for $|f| \le 2$. Therefore $R_X(f+\frac{k}{2})$ is non-zero for $-2f-4 \le k \le -2f+4$. Hence, for $0 < f \le \frac{1}{4}$ we have that $R_Z(f) = \sum_{k=-2f-4}^{-1}(2-|f+\frac{k}{2}|) + (2-|f|) + \sum_{k=1}^{-2f+4}(2-|f+\frac{k}{2}|) = \sum_{k=1}^{2f+4}(2+f-\frac{k}{2}) + (2-|f|) + \sum_{k=1}^{-2f+4}(2-f-\frac{k}{2}) = 8$. The result is similar for $-\frac{1}{4} < f \le 0$. Thus $R_Z(f) = \begin{cases} 8 & \text{if } -\frac{1}{4} < f \le \frac{1}{4} \\ 0 & \text{o.w.} \end{cases}$. □

3. Assume that $\{M_t : t > 0\}$ and $\{N_t : t > 0\}$ are independent Poisson processes both with intensity λ.

   (a) Is the process $\{M_t - N_t : t > 0\}$ weakly stationary? Prove your claim. (4p)

   (b) Calculate $P(M_{1.1} + N_{1.2} > 1)$ if $\lambda = 0.3$. (3p)

Solution:

(a) $C(M_s - N_s, M_t - N_t) = C(M_s, M_t) + C(N_s, N_t) \overset{s \leq t}{=} C(M_s, M_t - M_s) + C(M_s, M_s) + C(N_s, N_t - N_s) + C(N_s, N_s) = 0 + \lambda s + 0 + \lambda s = 2\lambda s$. The reasoning is similiar when $s > t$ resulting in $C(M_s - N_s, M_t - N_t) = 2\lambda t$ meaning that in general $C(M_s - N_s, M_t - N_t) = 2\lambda \min(s, t)$ which is not a function of $|s - t|$. Thus the process $\{M_t - N_t : t > 0\}$ is not weakly stationary.

(b) $\{M_t\} \perp \{N_t\} \Rightarrow M_{1.1} + N_{1.2} \in Poi(1.1 \cdot 0.3 + 1.2 \cdot 0.3) = Poi(0.69) \Rightarrow P(M_{1.1} + N_{1.2} > 1) = 1 - (P(M_{1.1} + N_{1.2} = 0) + P(M_{1.1} + N_{1.2} = 1)) = = 1 - \frac{0.69^0}{0!} e^{-0.69} - \frac{0.69^1}{1!} e^{-0.69} = 0.1523$. □

4. Let $\{\xi_t : t \in \mathbb{R}\}$ be a stationary Gaussian process with cvf $r_\xi(\tau) = e^{-\tau^2} \cos \tau$. Calculate

   (a) $P(\xi_t - \xi_{t+1} < 1)$. (3p)

   (b) the cvf of the derivative process $\{\xi'_t : t \in \mathbb{R}\}$. (3p)

Solution:

(a) $\{\xi_t\}$ Gaussian $\Rightarrow \xi_t - \xi_{t+1} \in N(\mu, \sigma^2)$ where $\mu = E(\xi_t - \xi_{t+1}) = 0$ and $\sigma^2 = C(\xi_t - \xi_{t+1}, \xi_t - \xi_{t+1}) = r_\xi(t-t) - r_\xi(t-(t+1)) - r_\xi(t+1-t) + r_\xi(t+1-(t+1)) = 2r_\xi(0) - 2r_\xi(1) = 2 - 2e^{-1}\cos(1) = 0.1987 \Rightarrow P(\xi_t - \xi_{t+1} < 1) = = \Phi(\frac{1-0}{\sqrt{0.1987}}) = 0.9875$.

(b) $r_{\xi'}(\tau) = -r''_\xi(\tau) = -\frac{d^2}{d\tau^2} e^{-\tau^2} \cos \tau = \frac{d}{d\tau} e^{-\tau^2}(2\tau \cos \tau + \sin \tau) = = e^{-\tau^2}\Big((2 - 4\tau^2)\cos \tau - 4\tau \sin \tau\Big)$. □

5. Calculate

$$\int_{\mathbb{R}} \frac{1}{x} \sum_{k=1}^{\infty} \delta_{2^k}(x)\, dx \qquad (4p)$$

Solution: By the definition of the delta function $\int_a^b f(x)\delta_c(x)\,dx = f(c)$ if $c \in [a, b]$ and $= 0$ o.w. Thus $\int_{\mathbb{R}} \frac{1}{x} \sum_{k=1}^{\infty} \delta_{2^k}(x)\,dx = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{1}{x} \delta_{2^k}(x)\,dx = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. □