①

$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$

Linjärt: $x(t) = a x_1(t) + b x_2(t)$
$y(t) = a \cdot y_1(t) + b \cdot y_2(t)$

Tidsinv: $x(t) \to y(t)$; $x(t-1) \to y(t-1)$

$$x(t) = u(t) \xrightarrow{A} y(t) = 0.8(1 - e^{-2t})\,u(t)$$

$$x(t) = \cos(2t) \xrightarrow{B} y(t) = 0{,}5657 \cos(2t - 0{,}7854)$$

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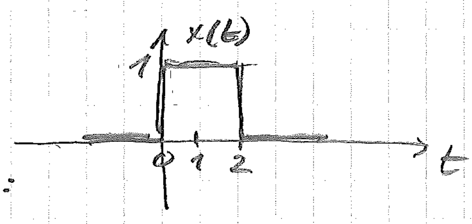
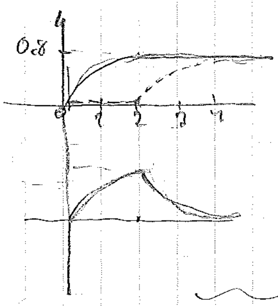
a) $$x(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t \leq 2 \\ 0 & t \geq 2 \end{cases}$$





Skriv $x(t)$ mha stegfunktionen $u(t)$:

$x(t) = u(t) - u(t-2)$

$$\underset{LTI}{\overset{A}{\Longrightarrow}} \; y(t) = 0.8(1 - e^{-2t}) \cdot u(t) - 0.8(1 - e^{-2(t-2)})\,u(t-2)$$

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b) $$x(t) = 3 \cos(2t - \pi/16)$$

$$\underset{LTI}{\overset{B.}{\Longrightarrow}} \; y(t) = 3 \cdot 0{,}5657 \cos(2t - \pi/16 - 0.7854)$$

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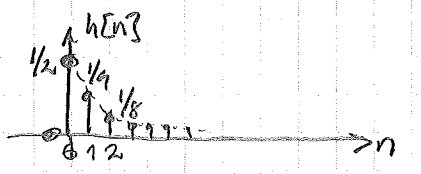
③ $$h[n] = 0.5(0.5)^n \cdot u[n]$$



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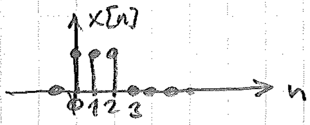
a) $x[n] = u[n] - u[n-3] = \{\underset{\uparrow}{1}, 1, 1\}$



$y[n] = h[n] * x[n]$; $n = 0, 1, 2, 3, 4, 5$

Grafisk lösning: $y[n] = \sum_k h(k)\, x[n-k]$ $n = 0, 1, 2, 3, 4, 5$

1/2	1/4	1/8	1/16	1/32	1/64	·	$h[k]$

1	1	1	→	$x[n-k]$

$y[0] = 1/2$
$y[1] = 1/2 + 1/4 = 3/4$
$y[2] = 1/2 + 1/4 + 1/8 = 7/8$
$y[3] = 1/4 + 1/8 + 1/16 = 7/16$
$y[4] = 1/8 + 1/16 + 1/32 = 7/32$
$y[5] = 1/16 + 1/32 + 1/64 = 7/64$

Matematisk lösning:

$y[n] = \sum_k x(k)\, h(n-k)$

$0 \leq n \leq 2$: $y[n] = \sum_{k=0}^{n} h(n-k)$
$= -0.5(0.5)^n(1 - 2^{n+1})$

$n \geq 3$: $y[n] = \sum_{k=0}^{2} h(n-k)$
$= 0.5(0.5)^n \cdot 7$

③ forts.

b)

$$h[n] = 0.5\,(0.5)^n \cdot u[n]$$

$$\mathcal{Z} \downarrow$$

$$H(z) = 0.5 \cdot \frac{1}{(1-0.5z^{-1})} = \frac{0.5}{(1-0.5z^{-1})} = \frac{Y(z)}{X(z)}$$

$$Y(z)(1-0.5z^{-1}) = 0.5 \cdot X(z)$$

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 \cdot X(z)$$

$$\mathcal{Z}^{-1} \downarrow$$

$$\boxed{y[n] - 0.5\,y[n-1] = 0.5\,x[n]}$$ diff. ekv.

$$y[0] = 0.5\,\underbrace{y[-1]}_{=0} + 0.5 \cdot 1 = 1/2$$
$$y[1] = 0.5\,y[0] + 0.5 \cdot 1 = 1/4 + 1/2 = 3/4$$
$$y[2] = 0.5\,y[1] + 0.5 \cdot 1 = 3/8 + 1/2 = 7/8$$
$$y[3] = 0.5\,y[2] + 0.5 \cdot 0 = 7/16$$
$$y[4] = 0.5\,y[3] + 0 = 7/32$$
$$y[5] = 0.5\,y[4] + 0 = 7/64$$

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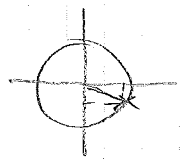
c)

$$x[n] = 1.5\cos\left(\frac{\pi}{6} n - \frac{\pi}{8}\right)$$

$$y[n]_{\text{stationär}} = \left|H\left(\tfrac{\pi}{6}\right)\right| 1.5 \cdot \cos\left(\frac{\pi}{6} n - \frac{\pi}{8} + \angle H\left(\tfrac{\pi}{6}\right)\right)$$

$$\boxed{H(\Omega) = H(z)\Big|_{z=e^{j\Omega}}} = \frac{0.5}{(1-0.5e^{-j\Omega})}$$

$z = e^{j\Omega} \in$ ROC av $H(z)$



$$H\left(\frac{\pi}{6}\right) = \frac{0.5}{(1-0.5e^{-j\pi/6})} = \frac{0.5}{\left(1-0.5\left(\frac{\sqrt{3}}{2} - j0.5\right)\right)} =$$

$$= \frac{0.5}{\left(1-\frac{\sqrt{3}}{4}\right) + j\frac{1}{4}} = 0.7383 - j0.3255$$

$$\begin{cases} \left|H\left(\frac{\pi}{6}\right)\right| = 0.8069 \\ \angle H\left(\frac{\pi}{6}\right) = -0.4153 \end{cases}$$

$$y[n]_{\text{stationär}} = 0.8069 \cdot 1.5\cos\left(\frac{\pi}{6} n - \frac{\pi}{8} - 0.4153\right)$$

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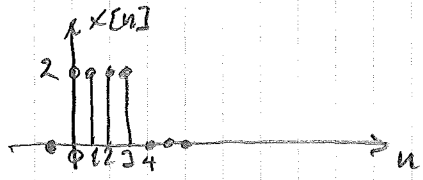
④ $x[n] = \begin{cases} 0 & n<0 \\ 2 & 0 \le n \le 3 \\ 0 & n \geq 4 \end{cases}$



a)

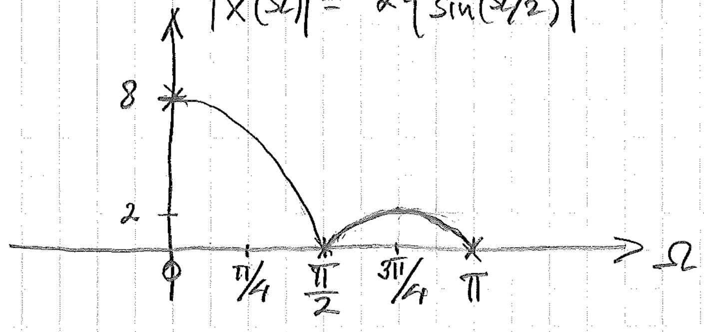
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{3} 2 \cdot e^{-j\Omega n} = 2 \sum_{n=0}^{3} \left(e^{-j\Omega}\right)^n$$

$$= \begin{cases} 2 \cdot \dfrac{1-e^{-j\Omega 4}}{1-e^{-j\Omega}} & \text{om } e^{-j\Omega} \neq 1 \Leftrightarrow \underline{\underline{\Omega \neq 0}} \\ 2 \cdot 4 & \text{om } \underline{\underline{\Omega = 0}} \end{cases}$$

$$= \begin{cases} 2 \cdot \dfrac{e^{-j\Omega 2}\left(e^{j\Omega 2} - e^{-j\Omega 2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)} = e^{-j\frac{3}{2}\Omega} \underbrace{2 \cdot \dfrac{\sin(2\Omega)}{\sin(\Omega/2)}}_{X_{reell}(\Omega)} \\ 8 \end{cases}$$

b) $|X(\Omega)| = 2 \cdot \left| \dfrac{\sin(2\Omega)}{\sin(\Omega/2)} \right|$



Nollställen?

$2\Omega = k \cdot \pi$

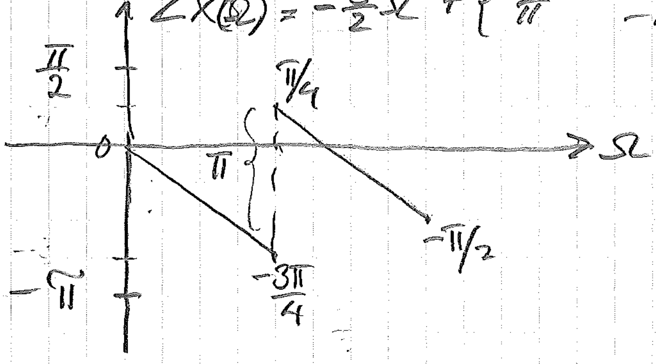
$k = \cancel{0}, 1, 2, 3, 4,$

$\Omega = k \cdot \pi/2$

+ sinc-form

$\left|X\left(\frac{3\pi}{4}\right)\right| \approx 2{,}16$

$\angle X(\Omega) = -\frac{3}{2}\Omega + \begin{cases} 0 & X_{reell}(\Omega) > 0 \\ \pi & X_{reell}(\Omega) < 0 \end{cases}$



(5)

$$\mathcal{L}\left\{ \boxed{\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = \dot{x}(t) + x(t)} \right.$$

a)

$$s^2 Y(s) + 4 \cdot s \cdot Y(s) + 8 \cdot Y(s) = s \cdot X(s) + X(s)$$

$$Y(s)\left(s^2+4s+8\right) = X(s)(s+1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+1)}{(s^2+4s+8)}$$

$$= \frac{(s+1)}{(s-p_1)(s-p_2)} \overset{pbu}{=} \frac{c_1}{(s-p_1)} + \frac{\bar{c}_1}{(s-\bar{p}_1)}$$

$$s^2+4s+8=0$$

$$p_{1,2} = -2 \pm \sqrt{\underbrace{4-8}_{-4}} = -2 \pm j2 \qquad \begin{cases} p_1 = -2+j2 = \sigma + j\omega \\ p_2 = -2-j2 \end{cases}$$

$$\begin{cases} c_1 = (s-p_1) \cdot H(s)\big|_{s=p_1} = \dfrac{p_1+1}{(p_1-p_2)} = \dfrac{-1+j2}{4j} \cdot \dfrac{j}{j} = -\dfrac{1}{4}(-2-j) \\ \qquad = \dfrac{1}{2} + \dfrac{1}{4}j \\ \qquad = \dfrac{\sqrt{5}}{4} e^{j0.4636} \\ \bar{c}_1 = \dfrac{1}{2} - \dfrac{1}{4}j \end{cases}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 2 \cdot |c_1| \cdot e^{\sigma t} \cdot \cos(\omega t + \angle c_1) \cdot u(t)$$

$$= 2 \cdot \frac{\sqrt{5}}{4} e^{-2t} \cos(2t + 0.4636) \cdot u(t)$$

b)

$$x(t) = u(t) \quad ; \quad X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$Y(s) = H(s) \cdot X(s) = \frac{(s+1)}{(s-p_1)(s-\bar{p}_1)} \cdot \frac{1}{s} \overset{pbu}{=} \frac{c_1}{(s-p_1)} + \frac{\bar{c}_1}{(s-\bar{p}_1)} + \frac{c_2}{s}$$

$$\begin{cases} c_1 = (s-p_1) \cdot Y(s)\big|_{s=p_1} = \dfrac{(p_1+1)}{(p_1-\bar{p}_1)} \dfrac{1}{p_1} = \dfrac{(-1+j2)}{4j(-2+j2)} \\ \qquad = \dfrac{(-1+j2)}{-8-8j} = \dfrac{1}{8} \dfrac{(-1+j2)}{(-1-j)} = 0{,}1976 e^{-j1.8925} \\ c_2 = s \cdot Y(s)\big|_{s=0} = \dfrac{1}{(-p_1)(-\bar{p}_1)} = \dfrac{1}{p_1 \bar{p}_1} = \dfrac{1}{|p_1|^2} = \dfrac{1}{8} \end{cases}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left(2 \cdot |c_1| e^{\sigma t} \cos(\omega t + \angle c_1) + \frac{1}{8}\right) u(t)$$

$$= \left(2 \cdot 0{,}1976 e^{-2t} \cos(2t - 1{,}8925) + \frac{1}{8}\right) u(t).$$

⑤ forts.

c) Slutvärdessatsen:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s)$$

$$\lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} \frac{s+1}{(s-p_1)(s-\bar{p}_1)} = \lim_{s\to 0} \frac{(s+1)}{(s^2+4s+8)} = \frac{1}{8}$$ jfr $y(t)$ i b)

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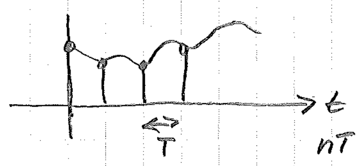
②

$$x(t) = \cos(2t) + \cos(20t - \pi/6)$$

$\omega_1 = 2$ rad/s $\qquad \omega_2 = 20$ rad/s.



$$\omega_s = 15 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega_s} = \frac{2\pi}{15} \text{ sek.}$$

Sampling: $t \to n\cdot T \qquad (\Omega = \omega\cdot T)$

$\Rightarrow$

$$x[n] = \cos\left(2\cdot\frac{2\pi}{15}\cdot n\right) + \cos\left(20\cdot\frac{2\pi}{15}\cdot n - \pi/6\right)$$

$$= \cos\left(\frac{4\pi}{15}\cdot n\right) + \cos\left(\frac{40\pi}{15}\cdot n - \frac{\pi}{6}\right)$$

$$\Omega_1 = \frac{4\pi}{15} \qquad \Omega_2 = \frac{40\pi}{15} = \frac{10\pi}{15} + \underbrace{\frac{30\pi}{15}}_{=2\pi} \overset{\text{alias!}}{=} \frac{2\pi}{3}$$

$$= \cos\left(\frac{4\pi}{15}\cdot n\right) + \cos\left(\frac{2\pi}{3}\cdot n - \frac{\pi}{6}\right)$$