Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
October 30, 2010, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove the sufficiency part of the *Martingale criterion for absence of arbitrage.* (5p)

   Solution: (Se p 413, *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Let $\{X_n : n \in \mathbb{Z}\}$ be an $MA(\infty)$ process with parameters $\mu = 0$ and $b_k = 2^{-k}$ for $k = 0, 1, 2, 3, \ldots$

   (a) Determine the variance of X_n, $D(X_n)$. (4p)

   (b) Calculate $P(X_{n+1} \leq 1 \,|\, X_n = 1)$. (3p)

   (c) Assuming $b_0 = 1$, $b_1 = \frac{1}{2}$ and $b_k = 0$ for $k = 2, 3, 4, \ldots$ Prove that the process $\{X_n\}$ is mesokurtic. (5p)

   Solution:

   (a) $X_n = 0 + 2^0\epsilon_n + 2^{-1}\epsilon_{n-1} + 2^{-n}\epsilon_{n-2} + \ldots$
   Since $\{\epsilon_n\}$ is white noise, i.e. independent and $N(0,1)$, we have that
   $D(X_n) = 1 \cdot D(\epsilon_n) + \frac{1}{2^2}D(\epsilon_{n-1}) + \frac{1}{2^4}D(\epsilon_{n-2}) + \ldots = \sum_{k=0}^{\infty}\frac{1}{2^{2k}} = \sum_{k=0}^{\infty}(\frac{1}{4})^k = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

   (b) $X_{n+1} = \epsilon_{n+1} + \frac{1}{2}\epsilon_n + \frac{1}{2^2}\epsilon_{n-1} + \frac{1}{2^3}\epsilon_{n-2} + \ldots = \epsilon_{n+1} + \frac{1}{2}(\epsilon_n + \frac{1}{2}\epsilon_{n-1} + \frac{1}{2^2}\epsilon_{n-2} + \ldots) = \epsilon_{n+1} + \frac{1}{2}X_n$
   $\Rightarrow\ P(X_{n+1} \leq 1 \,|\, X_n = 1) = P(\epsilon_{n+1} + \frac{1}{2}\cdot 1 \leq 1) = P(\epsilon_{n+1} \leq \frac{1}{2}) = \Phi(0.5) = 0.6915$.

   (c) To prove that $\{X_n\}$ is mesokurtic we have to show that the kurtosis

$$\gamma_2 \;=\; \frac{E((X_n-\mu)^4)}{(E((X_n-\mu)^2))^2} - 3 = 0$$

Since $\mu = 0$ and $b_k = 0$ for $k \geq 2$ we have that $X_n = \epsilon_n + \frac{1}{2}\epsilon_{n-1}$ so $E(X_n^4) = E((\epsilon_n + \frac{1}{4}\epsilon_{n-1})^4) = E(\epsilon_n^4 + 4 \cdot \frac{1}{2}\epsilon_n^3\epsilon_{n-1} + 6(\frac{1}{2})^2\epsilon_n^2\epsilon_{n-1}^2 + 4(\frac{1}{2})^3\epsilon_n\epsilon_{n-1}^3 + (\frac{1}{2})^4\epsilon_{n-1}^4) = E(\epsilon_n^4) + 2E(\epsilon_n^3)E(\epsilon_{n-1}) + \frac{3}{2}E(\epsilon_n^2)E(\epsilon_{n-1}^2) + \frac{1}{2}E(\epsilon_n)E(\epsilon_{n-1}^3) + \frac{1}{16}E(\epsilon_{n-1}^4) = 3 + 0 + \frac{3}{2} + 0 + \frac{1}{16} \cdot 3 = \frac{75}{16}$. Further $E(X_n^2) = 1^2 + (\frac{1}{2})^2 = \frac{5}{4}$ so $E(X_n^2)^2 = \frac{25}{16}$ and thus $\gamma_2 = \frac{75/16}{25/16} - 3 = 0$, i.e. $\{X_n\}$ is mesokurtic. □

3. Calculate the fourth moment $E(h_n^4)$ of an $ARCH(1)$ process $\{h_n\}$. (4p)

Solution: For the $ARCH(1)$ process $h_n = \sigma_n\epsilon_n$ where $\sigma_n^2 = a_0 + a_1h_{n-1}^2$. Since it is a weakly stationary process we have that $E(h_n^4) = E(h_{n-1}^4)$, a constant we may denote by m_4, and $m_2 = E(h_{n-1}^2) = E(h_{n-2}^2)$ and due to independence between ϵ_n and σ_n we get

$$\begin{aligned} m_4 &= E(h_n^4) \\ &= E(\sigma_n^4)E(\epsilon_n^4) \\ &= 3E((a_0 + a_1h_{n-1}^2)^2) \\ &= 3(a_0^2 + a_1^2E(h_{n-1}^4) + 2a_0a_1E(h_{n-1}^2)) \end{aligned}$$

Now, $m_2 = E(h_n^2) = E(\sigma_n^2)E(\epsilon_n^2) = E(a_0 + a_1h_{n-1}^2) = a_0 + a_1E(h_{n-1}^2) \Rightarrow m_2 = \frac{a_0}{1-a_1}$. Therefore $m_4(1 - 3a_1^2) = 3a_0^2 + 6a_0a_1\frac{a_0}{1-a_1}$ which means that

$$\begin{aligned} E(h_n^4) &= m_4 \\ &= \frac{3a_0^2 + \frac{6a_0^2a_1}{1-a_1}}{1 - 3a_1^2} \\ &= \frac{3a_0^2(1-a_1) + 6a_0^2a_1}{(1-3a_1^2)(1-a_1)} \\ &= \frac{3a_0^2(1+a_1)}{(1-3a_1^2)(1-a_1)} \end{aligned}$$

□

4. Let $\{h_n : n \in \mathbb{Z}^+\}$ be an asymetric random walk such that $h_0 = 0$ and $h_n = \sum_{k=1}^{n} U_k$ for $n = 1, 2, 3, \ldots$ where the sequence $\{U_k\}$ consists of independent random variables such that $P(U_k = 1) = 1 - P(U_k = C) = p$ where $0 < p < 1$.
   (a) Determine the number C such that $\{h_n\}$ is a martingale with respect to the flow $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(U_1, U_2, \ldots, U_n)$. (4p)
   (b) Assume that $C = -1$ and that the observations $h_1 = 1$, $h_2 = 2$, $h_3 = 1$, $h_4 = 2$, $h_5 = 3$ are made. Calclulate the maximum likelihood estimator of p. (5p)

Solution:

(a) For the condition $E(h_{n+1}\,|\,\mathcal{F}_n)$ to be satisfied we have to have $E(h_{n+1}\,|\,\mathcal{F}_n) = E(U_{n+1} + h_n\,|\,\mathcal{F}_n) = h_n + E(U_{n+1}) = h_n$, i.e. $E(U_{n+1})$ should equal 0. Thus $E(U_{n+1}) = 1 \cdot p + C(1-p) = 0 \;\Rightarrow\; p + C - Cp = 0 \;\Rightarrow\; C(1-p) + p = 0 \;\Rightarrow\; C = -\frac{p}{1-p}$ (ok, since $p < 1$).

(b) Assume $C = -1$. Since $h_0 = 0$ we have that $P(h_1 = 1, h_2 = 2, h_3 = 1, h_4 = 2, h_5 = 3) = P(h_1 = 1)P(h_2 = 2\,|\,h_1 = 1)P(h_3 = 1\,|\,h_2 = 2)P(h_4 = 2\,|\,h_3 = 1)P(h_5 = 3\,|\,h_4 = 2) = p \cdot p \cdot (1-p) \cdot \cdot p = p^5 - p^4 = \ell(p)$. The value of p which maximizes ℓ should satisfy $\ell'(p) = 0$, i.e. $p^3(4 - 5p) = 0$. Since $p > 0$ the only solution is $p = \frac{4}{5} = 0.8$. Thus the value of the estimator is $\hat{p} = 0.8$ in this case. $\square$