

Multivariable calculus, 2009-10-30.

Per-Sverre Svendsen, Tel.035 - 167 615/0709 - 398 526.

Minimum requirements: Grade 5: 27p, 4: 21p, 3: 15p.

You are allowed to use one unmarked, ordinary (non-mathematical) dictionary of your choice. No additional written material is allowed. Also, no calculators or any other electronic equipment.

1. Calculate the directional derivative of $f(x, y, z) = \ln\left(\frac{z}{x^2 + y^2}\right)$ at the point $P = (1, 2, 5)$ and in the direction $\mathbf{v} = (2, -1, 2)$. (2p)

2. Find the following limit or show that it does not exist $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2 - 2xy)}{x^2 + y^2}$. (2p)

3. The equation $x^2z^3 + 2xz - \frac{y}{z} + 1 = 0$ implicitly defines a function $z = g(x, y)$ for which $g(1, 2) = -1$. Calculate $g_x(1, 2)$ and $g_y(1, 2)$. (2p)

4. Find the center of mass $P = (\bar{x}, \bar{y})$ of a region D with constant density, where $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y \leq 1\}$. (3p)

Hint: $\bar{x} = \frac{1}{A_D} \iint_D x \, dx \, dy$, $\bar{y} = \frac{1}{A_D} \iint_D y \, dx \, dy$, where A_D is the area of the region D .

5. Calculate the area of the region in the first quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$. (3p)

6. Find the absolute minimum and maximum values of $f(x, y, z) = x + 2y - 3z$ subject to the constraint $x^2 + 4y^2 + 9z^2 = 108$. (3p)

7. Find the absolute minimum and maximum values of $g(x, y) = 9xy^2e^{-2x-3y}$ on the set $\Delta = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$. (5p)

8. Calculate $\iint_S x^2z^2 \, dS$, where $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2}, 1 \leq x^2 + y^2 \leq 2\}$. (5p)

9. Calculate $\iiint_K \frac{z}{1 + \sqrt{x^2 + y^2 + z^2}} \, dx \, dy \, dz$, $K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}$. (5p)

Reference formulas and equations in Multivariable Calculus

Trigonometry and Logarithms

$$\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y & \sin x \pm \sin y &= 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2} \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y & \cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} & \cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \\ \cot(x \pm y) &= \frac{\cot x \cot y \mp 1}{\pm \cot x + \cot y} & 2 \sin x \sin y &= \cos(x-y) - \cos(x+y) \\ \sin 2x &= 2 \sin x \cos x & 2 \cos x \cos y &= \cos(x-y) + \cos(x+y) \\ \cos 2x &= \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x & 2 \sin x \cos y &= \sin(x-y) + \sin(x+y) \\ \ln x + \ln y &= \ln xy & \ln x - \ln y &= \ln \frac{x}{y} \\ \ln x^a &= a \ln x & & (x, y > 0) \end{aligned}$$

Standard limits

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^\alpha \log_a x &= 0 \quad (a > 1, \alpha > 0) & \lim_{x \rightarrow \infty} \frac{a^x}{x^\alpha} &= \infty \quad (a > 1) \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} &= 1 & \lim_{x \rightarrow \infty} \frac{x^\alpha}{\log_a x} &= \infty \quad (a > 1, \alpha > 0) \\ \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} &= 1 & \lim_{n \rightarrow \infty} \frac{a^n}{n!} &= 0 \\ \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= 1 & & \end{aligned}$$

Basic derivatives

$f(x)$	$f'(x)$
x^a	ax^{a-1}
a^x	$a^x \ln a$
$\ln x $	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\ln x + \sqrt{x^2 + \alpha} $	$\frac{1}{\sqrt{x^2 + \alpha}}$
$\frac{1}{2}x\sqrt{x^2 + \alpha} + \frac{\alpha}{2} \ln x + \sqrt{x^2 + \alpha} $	$\sqrt{x^2 + \alpha}$

Taylor Series

Taylor's formula for a function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n$$

Table of particular expansions ($a = 0$, $h \rightarrow x$)

1. $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$ ($-1 < x < 1$)
2. $(x+1)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2 \cdot 3}x^3 + \dots$ ($-1 < x < 1$)
3. $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$
4. $\sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$
5. $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$
6. $\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$ ($-1 < x \leq 1$)
7. $\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} x^{2k-1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$ ($-1 \leq x \leq 1$)

Taylor's formula for a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(a+h, b+k) = f(a, b) + hf_x(a, b) + kf_y(a, b) + \frac{1}{2}(h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)) + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x, y)_{(a,b)}$$

Tangent plane

Function $z = f(x, y)$

Equation of tangent plane through the point $(a, b, f(a, b))$

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

Level surface $F(x, y, z) = C$

Equation of tangent plane through the point (a, b, c)

$$F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$$

Directional derivative

The directional derivative of a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ at the point (a, b, c) and direction \mathbf{u} ($|\mathbf{u}| = 1$)

$$D_{\mathbf{u}}f(a, b, c) = f'_{\mathbf{u}}(a, b, c) = \mathbf{u} \cdot \nabla f(a, b, c) = \mathbf{u} \cdot (f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)).$$

Double Integrals

General substitution

Assume a one-to-one mapping between a region D in the xy -plane and a region D_{uv} in the uv -plane

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} \Leftrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

$$\text{Then } \iint_D f(x, y) \, dx dy = \iint_{D_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du dv,$$

$$\text{with } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0.$$

Polar coordinates

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \quad \iint_D f(x, y) \, dx dy = \iint_{D_{r\theta}} f(r \cos \theta, r \sin \theta) \, r dr \, d\theta$$

Triple Integrals

General substitution

As above assume a one-to-one mapping between points (x, y, z) in Δ and (u, v, w) in Δ_{uvw} .

$$\iiint_{\Delta} f(x, y, z) \, dx dy dz = \iiint_{\Delta_{uvw}} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du dv dw,$$

$$\text{with } \frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0$$

Spherical coordinates

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi$$

$$\iiint_{\Delta} f(x, y, z) \, dx dy dz = \iiint_{\Delta_{\rho\phi\theta}} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Surface Integrals

General parametrized surface

$S : \mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in D.$

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv$$

Function graph $z = h(x, y)$

$S : \mathbf{r} = (x, y, z) = (x, y, h(x, y)), (x, y) \in S_{xy}.$

$$\iint_S f(x, y, z) dS = \iint_{S_{xy}} f(x, y, h(x, y)) \sqrt{1 + h_x^2 + h_y^2} dx dy$$