

## Solutions Multivariable calculus, 2010-10-29.

1. Find an equation of the tangent plane to the surface  $x^3 - zy^2 + 2z^3 + 1 = 0$  at the point  $(1, 2, 1)$ . (2p)

*Answer:*

Setting  $F(x, y, z) = x^3 - zy^2 + 2z^3 + 1$ , the surface above can be viewed as the level surface  $F(x, y, z) = 0$ . A normal vector,  $\mathbf{n}$ , to the tangent plane is then:

$$\mathbf{n} = \nabla F(x, y, z)_{(1,2,1)} = (3x^2, -2zy, -y^2 + 6z^2)_{(1,2,1)} = (3, -4, 2).$$

This gives the equation of the tangent plane:

$$\mathbf{n} \cdot (x - 1, y - 2, z - 1) = 0 \Leftrightarrow 3(x - 1) + (-4)(y - 2) + 2(z - 1) = 0 \Leftrightarrow 3x - 4y + 2z + 3 = 0.$$

2. Find the following limit or show that it does not exist  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$ . (2p)

*Answer:*

Set :  $x = r \cos \theta$ ,  $y = r \sin \theta$ . This gives :

$$\frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \frac{r^2(\cos^2 \theta - \sin^2 \theta)}{r} = r \cos 2\theta.$$

Now,  $|\cos 2\theta| \leq 1$ , and  $r \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ . Therefore:  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = 0$ .

3. Find and classify all critical points of the function  $f(x, y) = x^2 + y^2 - 6xy + 5x + 9y$ . (2p)

*Answer:*

We first find all critical points (as it turns out, there is only one):

$$\begin{aligned} f_x = 0 &\Leftrightarrow 2x - 6y + 5 = 0 \\ f_y = 0 &\Leftrightarrow -6x + 2y + 9 = 0 \end{aligned} \Leftrightarrow (x, y) = \left(2, \frac{3}{2}\right).$$

Also, we need the 2nd partial derivatives:  $f_{xx} = 2$ ,  $f_{yy} = 2$ ,  $f_{xy} = -6$ .

An expansion to 2nd order around the point  $(2, \frac{3}{2})$  then yields:

$$f\left(2+h, \frac{3}{2}+k\right) \simeq f\left(2, \frac{3}{2}\right) + \frac{1}{2}\left(2h^2 + 2 \cdot (-6) \cdot hk + 2k^2\right) = \frac{47}{4} + h^2 - 6hk + k^2 = \frac{47}{4} + (h-3k)^2 - 8k^2.$$

From this expression we see that that  $(2, \frac{3}{2})$  is a saddle-point.

4. Calculate  $\iint_D x^2 e^{xy} dx dy$ , where  $D$  is a triangle with vertices at  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$ . (3p)

*Answer:*

$$\begin{aligned} \iint_D x^2 e^{xy} dx dy &= \int_0^1 x^2 \left( \int_0^x e^{xy} dy \right) dx = \int_0^1 x^2 \left[ \frac{1}{x} e^{xy} \right]_0^x dx = \int_0^1 x (e^{x^2} - 1) dx = \\ &= \left[ \frac{1}{2} e^{x^2} - \frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} (e - 2). \end{aligned}$$

5. Calculate  $\iint_{\Delta} \frac{(2x+y)^2}{(2x-y)^2+1} dx dy$ ,  
 where  $\Delta = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq 2x-y \leq 1, 0 \leq 2x+y \leq 2\}$ . (3p)

*Answer:*

The parallelogram  $\Delta$  in the  $xy$ -plane can be transformed into a rectangle  $\Delta_{uv} : 0 \leq u \leq 1, 0 \leq v \leq 2$  by the substitution:

$$u = 2x - y, \quad v = 2x + y, \quad \text{which gives } x = \frac{1}{4}u + \frac{1}{4}v, \quad y = -\frac{1}{2}u + \frac{1}{2}v, \quad \text{and } \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}.$$

Inserted into the integral:

$$\begin{aligned} \iint_{\Delta} \frac{(2x+y)^2}{(2x-y)^2+1} dx dy &= \int_0^2 \int_0^1 \frac{v^2}{u^2+1} \left| \frac{1}{4} \right| du dv = \frac{1}{4} \int_0^2 v^2 dv \int_0^1 \frac{1}{u^2+1} du = \\ &= \frac{1}{4} \left[ \frac{1}{3}v^3 \right]_0^2 \left[ \arctan u \right]_0^1 = \frac{\pi}{6}. \end{aligned}$$

6. Find the absolute minimum and maximum values of  $f(x,y,z) = 3x - 2y - z$  subject to the constraint  $x^2 + y^2 + z^2 = 14$ . (3p)

*Answer:*

Using the method of Lagrange multipliers and letting  $g(x,y,z) = x^2 + y^2 + z^2 - 14 = 0$  represent the constraint :

$$\begin{aligned} f_x &= \lambda g_x & 3 &= 2\lambda x \\ f_y &= \lambda g_y & -2 &= 2\lambda y \Rightarrow 2\lambda x = (-3) \cdot 2\lambda z \\ f_z &= \lambda g_z & -1 &= 2\lambda z \Rightarrow 2\lambda y = 2 \cdot 2\lambda z \Rightarrow (x,y,z) = (-3z, 2z, z). \end{aligned}$$

This solution is then inserted into the constraint (note that  $\lambda = 0$  is inconsistent with the original Lagrange equations):

$$(-3z)^2 + (2z)^2 + z^2 = 14 \Leftrightarrow z = -1 \vee z = 1.$$

Comparison and conclusion:

$$f(3, -2, -1) = 14, \quad f(-3, 2, 1) = 14, \quad f_{min} = -14, \quad f_{max} = 14.$$

7. Find the absolute minimum and maximum values of  $g(x,y) = (x^2 + y^2) e^{-x^2 - 4y^2}$  on the set  $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \leq 4\}$ . (5p)

*Answer:*

First finding interior critical points:

$$\begin{aligned} g_x &= 0 & \Leftrightarrow & 2xe^{-x^2-4y^2}(1-x^2-y^2) = 0 \\ g_y &= 0 & \Leftrightarrow & 2ye^{-x^2-4y^2}(1-4x^2-4y^2) = 0 \\ (x=0 \vee x^2+y^2=1) \wedge (y=0 \vee 4x^2+4y^2=1) & \Leftrightarrow (x,y) = (0,0) \vee (x,y) = (0, \pm \frac{1}{2}) \vee (x,y) = (\pm 1, 0). \end{aligned}$$

All of these 5 points are interior to  $A$ .

Next we consider the values of  $g$  on the boundary of  $A$ , that is, the ellipse  $x^2 + 4y^2 = 4$ . We parametrize the boundary curve as:  $x = 2 \cos t, y = \sin t, 0 \leq t < 2\pi$ . Inserted:

$$g(2 \cos t, \sin t) = (4 \cos^2 t + \sin^2 t) e^{-4} = (3 \cos^2 t + 1) e^{-4}.$$

From the expression above, the maximum and minimum values on the boundary are seen to be  $g(2,0) = g(-2,0) = 4e^{-4}$  and  $g(0,1) = g(0,-1) = e^{-4}$ , respectively.

Comparing with the values at the interior critical points:

$$g(0,0) = 0, \quad g(0, \pm \frac{1}{2}) = \frac{e^{-1}}{4}, \quad g(\pm 1, 0) = e^{-1}.$$

Since  $e^{-1} > 4e^{-4}$ , we then have the following absolute extrema on  $A$ :

$$f_{min} = 0, \quad f_{max} = e^{-1}.$$

8. Calculate  $\iiint_K \frac{xyz}{x^2 + y^2 + z^2 + 1} dx dy dz$ ,

$$K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}. \quad (5p)$$

Answer:

Using spherical coordinates and noting that  $K$  is the 1st octant part of a sphere with radius  $R = 1$  we obtain the limits:

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \rho \leq 1.$$

The integral can then be calculated as:

$$\begin{aligned} \iiint_K \frac{xyz}{x^2 + y^2 + z^2 + 1} dx dy dz &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\rho \sin \phi \cos \theta \cdot \rho \sin \phi \sin \theta \cdot \rho \cos \phi}{\rho^2 + 1} \rho^2 d\rho \sin \phi d\phi d\theta = \\ &= \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \int_0^{\frac{\pi}{2}} (1 - \cos^2 \phi) \cos \phi \sin \phi d\phi \int_0^1 \frac{\rho^5}{\rho^2 + 1} d\rho = [u = \cos \phi] = \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin(2\theta) d\theta \int_0^1 (1 - u^2) u du \int_0^1 \left( \rho^3 - \rho + \frac{\rho}{\rho^2 + 1} \right) d\rho = \\ &= \frac{1}{2} \left[ -\frac{1}{2} \cos(2\theta) \right]_0^{\frac{\pi}{2}} \left[ \frac{1}{2} u^2 - \frac{1}{4} u^4 \right]_0^1 \left[ \frac{1}{4} \rho^4 - \frac{1}{2} \rho^2 + \frac{1}{2} \ln(\rho^2 + 1) \right]_0^1 = \frac{1}{32} (2 \ln 2 - 1). \end{aligned}$$

Alternatively, the integral can be calculated as (where  $K_{xy}$  is the projektion of  $K$  on to the  $xy$ -plane):

$$\begin{aligned} \iiint_K \frac{xyz}{x^2 + y^2 + z^2 + 1} dx dy dz &= \int_{K_{xy}} xy \left( \int_0^{\sqrt{1-x^2-y^2}} \frac{z}{x^2 + y^2 + z^2 + 1} dz \right) dx dy = \\ &= \int_{K_{xy}} xy \left[ \frac{1}{2} \ln(x^2 + y^2 + z^2 + 1) \right]_{z=0}^{\sqrt{1-x^2-y^2}} dx dy = \frac{1}{2} \iint_{K_{xy}} xy \left( \ln 2 - \ln(x^2 + y^2 + 1) \right) dx dy = \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^1 r \sin \theta r \cos \theta \left( \ln 2 - \ln(r^2 + 1) \right) r dr d\theta = \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin(2\theta) d\theta \left( \ln 2 \int_0^1 r^3 dr - \int_0^1 r^3 \ln(r^2 + 1) dr \right) = \frac{\ln 2}{16} - \frac{1}{4} \left( \left[ \frac{1}{4} r^4 \ln(r^2 + 1) \right]_0^1 - \frac{1}{4} \int_0^1 r^4 \frac{2r}{r^2 + 1} dr \right) = \\ &= \frac{\ln 2}{16} - \frac{\ln 2}{16} + \frac{1}{8} \left[ \frac{1}{4} r^4 - \frac{1}{2} r^2 + \frac{1}{2} \ln(r^2 + 1) \right]_0^1 = \frac{1}{32} (2 \ln 2 - 1). \end{aligned}$$

(The integral  $\int r^3 \ln(r^2 + 1) dr$  is carried out by a combination of *integration by parts* and the result above for  $\int \frac{\rho^5}{\rho^2 + 1} d\rho$ ).

9. Calculate  $\iint_S z \, dS$ , where the surface  $S$  is the part of the paraboloid

$$z = 2 - x^2 - y^2 \text{ that lies above the cone } z = \sqrt{x^2 + y^2}. \quad (5p)$$

*Answer:*

We first note that (with  $z = 2 - x^2 - y^2$ ):  $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2}$ .

Furthermore, the projection of the surface  $S$  on to the  $xy$  - plane is given by the condition

$$2 - x^2 - y^2 \geq \sqrt{x^2 + y^2} \Leftrightarrow r^2 + r - 2 \leq 0 \Rightarrow r \leq 1, \quad (r = \sqrt{x^2 + y^2}).$$

This means that the projection of  $S$  on to the  $xy$  - plane is a unit disk centered at the origin. The integral is then calculated as:

$$\iint_S z \, dS = \iint_{S_{xy}} (2 - x^2 - y^2) \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy =$$

$$\int_0^{2\pi} \int_0^1 (2 - r^2) \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \left[ u = 1 + 4r^2 \right] =$$

$$\frac{\pi}{16} \int_1^5 (9 - u) \sqrt{u} \, du = \frac{\pi}{16} \left[ 6u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_1^5 = \frac{\pi}{20} (25\sqrt{5} - 7).$$