Exam for Random Processes, 7.5 ECTS

January 13, 2012, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

For each problem a *complete* solution should be given. All solutions should be thoroughly presented and all mathematical expressions should be simplified as far as possible. Each solution should start at the top of a new sheet of paper. The proper solutions will be available on the internet at
`http://dixon.hh.se/erja` → `Teaching` → `Random processes` → `Previous exams`

1. Assume that $\{X_t\}$ is a weakly stationary process which is filtered through a linear filter with frequency function H resulting in the output signal $\{Y_t\}$. Prove that $R_Y(f) = |H(f)|^2 R_X(f)$. (3p)

2. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process with expectation function $m(t) = 1$ and covariance function $r(\tau) = \pi e^{-\pi|\tau|}\cos(\pi\tau)$. Calculate
   (a) $P(X_t + X_{t+1} < 1)$. (3p)
   (b) the spectral density of $\{X_t\}$. (3p)

3. Assume that the weakly stationary process $\{Y_t : t \in \mathbb{Z}\}$ is defined by the recursive relationship $Y_t = e_t + \frac{1}{2}e_{t-1} + \frac{1}{3}Y_{t-1}$ where $\{e_t : t \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables with $E(e_t) = 0$ and $V(e_t) = 2$.
   (a) What is a random process like $\{e_t\}$ called? (1p)
   (b) What is a random process like $\{Y_t\}$ called? (2p)
   (c) Calculate $C(Y_t, Y_{t-1})$. (4p)

4. Let $\{X_t : t \in \mathbb{R}\}$ be shot noise with intensity λ and and impulse function

$$g(t) = \begin{cases} (1-t)^2 & \text{when } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate the expectation function of $\{X_t\}$. (3p)
   (b) What values of λ makes the mean effect bounded by 1, i.e. what λ values makes $E(X_t^2) < 1$? (3p)

5. Calculate $V(\int_0^1 X_t dt)$ if $\{X_t : t \in \mathbb{R}\}$ is a weakly stationary process with covariance function $r(\tau) = 1 - \tau^2$ when $|\tau| < 1$ and 0 otherwise. (4p)

6. Let $f(x) = \sum_{j=1}^{\infty} \frac{\delta_{q_j}(x)}{j!}$ where $\{q_j : j \in \mathbb{Z}^+\}$ are the positive rational numbers $(\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$. Determine the value of $\int_{\mathbb{R}} f(x)dx$. (4p)

GOOD LUCK!

Some rules of Probability and Statistics

Def If an experiment has m outcomes of which b results in the event A, then the probability of A is

$$P(A) \;=\; g/m\,.$$

Def P is a probability measure if

1. $0 \le P(A) \le 1$
2. $P(\Omega) = 1$
3. $A \cap B = \emptyset \;\Rightarrow\; P(A \cup B) = P(A) + P(B)$

for all events $A \subset \Omega$ where Ω is the sample space.

Basic principle of counting If an experiment can be divided into j subexperiments where the
first has n_1 outcomes
second has n_2 outcomes
$\vdots$
jth has n_j outcomes
then the experiment totally has $n_1 \cdot n_2 \cdots n_j$ outcomes.

Addition theorem $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Def A and B are independent events if $P(A \cap B) = P(A)P(B)$

Def The conditional probability of A given B is $P(A|B) \;=\; \dfrac{P(A \cap B)}{P(B)}$

Bayes theorem If $A_1, \ldots, A_n$ is a partition of Ω
(i.e. $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i = \Omega$).
then $P(A_k|B) \;=\; \dfrac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$ for each $k = 1, 2, \ldots, n$.

Combinatorics The number of ways to choose k elements among n possible, without replacement and without respect to the order, is

$$\binom{n}{k} \;=\; \frac{n!}{k!\,(n-k)!} \;=\; \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1}$$

Random variables

X discrete: Probability fcn: $p(x) = P(X = x)$

Distribution fcn: $P(X \le a) = F(a) = \sum_{x \le a} p(x)$,

X cont.: Density fcn: $f(x) = \frac{d}{dx} P(X \le x)$

Distribution fcn: $P(X \le a) = F(a) = \int_{-\infty}^{a} f(x)\, dx$.

Expected value and variance

X discrete: The expected value of X: $\mu = E(X) = \sum_{x \in \Omega} x\, p(x)$.

Variance of X: $\sigma^2 = V(X) = \sum_{x \in \Omega} (x - \mu)^2\, p(x)$.

X cont: Expected value of X: $\mu = E(X) = \int_{x \in \Omega} x\, f(x)\, dx$.

Variance of X: $\sigma^2 = V(X) = \int_{x \in \Omega} (x - \mu)^2\, f(x)\, dx$.

Covariance of X and Y: $C(X, Y) = E((X - \mu_x)(Y - \mu_y))$

Correlation of X and Y: $\rho = \frac{C(X,Y)}{\sqrt{V(X)V(Y)}}$

Standarddev. $\sigma = \sqrt{V(X)}$.

Linearity: $E(aX + bY) = a\, E(X) + b\, E(Y)$ for all random variables X and Y and real numbers a and b.

If X, Y indep. then $V(aX + bY) = a^2 V(X) + b^2 V(Y)$.

Rules:

$E(g(X)) = \int_{\mathbb{R}} g(x)\, f(x)\, dx$

$V(X) = E(X^2) - (E(X))^2$

$C(X, Y) = \int\int_{\mathbb{R}^2} xy\, f(x, y) - E(X)E(Y)$

$C(\sum_i a_i X_i, \sum_k b_k Y_k) = \sum_i \sum_k a_i b_k C(X_i, Y_k)$

Normal distribution

denoted by $N(\mu, \sigma^2)$ where μ is the expected value and σ^2 is the variance

$N(0, 1)$ is called standard normal distribution with density function $\Phi(x)$

If $X \in N(\mu, \sigma^2)$ then $P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Symmetry: $\Phi(-x) = 1 - \Phi(x)$

Probabilities: $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

Def The random variables $X_1, X_2, \ldots, X_n$ are a sample of X if all variables, X_i, is distributed as X, $i = 1, \ldots, n$, and all variables are independent of each other at all levels.

CLT (Central Limit Theorem)

If $X_1, \ldots, X_n$ is a sample where $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, $i = 1, \ldots, n$

then $P\left(\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) \le x\right) \rightarrow \Phi(x)$ as $n \rightarrow \infty$.

This implies that $\sum_{i=1}^{n} X_i$ is approximately $N(n\mu, n\sigma^2)$ and $\bar{X}$ is approximately $N(\mu, \sigma^2/n)$ for large n.

Approximations

Distribution	Condition	Approximative distribution
$Bin(n, \pi)$	$n \ge 10$ and $\pi < 0.1$	$Po(n\pi)$
$Bin(n, p)$	$np(1-p) \ge 10$	$N(np, np(1-p))$
$Poi(\lambda)$	$\lambda \ge 15$	$N(\lambda, \lambda)$

Def $\{X_t\} = \{X_t : t \in T \subseteq \mathbb{R}\}$ is a **random process** if all variables $X_t, t \in T$ have the same probability distribution regardless of t. T is called **index space**.

Thm <u>If</u> $\{X_t\}$ has independent, stationary increments and $X_0 = 0$
<u>then</u> $r_X(s,t) = \min(s,t) \cdot V(X_1)$.

Random phase and amplitude

<u>If</u> $X_t = A_0 + \sum_{k=1}^{n} A_k \cos(2\pi f_k t + \phi_k)$
$\phi_k \in R(0, 2\pi)$
$\{A_k\}$ are independent and $\sigma_0 = E(A_0^2)$, $\sigma_k^2 = E(A_k^2)$

<u>then</u> $r_X(\tau) = \sigma_0^2 + \sum_{k=1}^{n} \frac{\sigma_k^2}{2} \cos(2\pi f_k \tau)$

Spectral density

Relationship between cvf, $r(\tau)$, and spectral density, $R(f)$,

Continuous time	Discrete time
$r(\tau) = \int_{\mathbb{R}} R(f) e^{i2\pi f\tau} df$	$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f\tau} df$
$R(f) = \int_{\mathbb{R}} r(\tau) e^{-i2\pi f\tau} d\tau$	$R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f\tau}$

Aliasing

<u>If</u> $\{X_t\}$ is a random process which is sampled into
$\{Y_k\} = \{X_{kd} : k = 0, \pm 1, \pm 2, \ldots\}$ (normed frequency)
$\{Z_t\} = \{X_t : t = 0, \pm d, \pm 2d, \ldots\}$ (sampled process)

<u>then</u> $r_Y(\frac{\tau}{d}) = r_Z(\tau) = r_X(\tau)$ for $\tau = 0, \pm d, \pm 2d, \ldots$
$R_Y(f) = \frac{1}{d} \sum_{k=-\infty}^{\infty} R_X(\frac{f+k}{d})$ for $f \in (-\frac{1}{2}, \frac{1}{2}]$
$R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f + \frac{k}{d})$ for $f \in (-\frac{1}{2d}, \frac{1}{2d}]$

Mulitvariate normal distribution

Two-dimensional (bivariate) normal distribution

<u>If</u> $E(X_1) = E(X_2) = 0$, $V(X_1) = V(X_2) = 1$ and $C(X_1, X_2) = \rho$

<u>then</u> $\boldsymbol{X} = [X_1\ X_2]^T \in N_2(\boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} = [0\ 0]^T$ and $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.
$f_{\boldsymbol{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big(- \frac{1}{2(1-\rho^2)}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\Big)$

Campbell's formulae

<u>If</u> $\{X_t\}$ is the shot-noise process with parameter λ and pulse-function $g(s)$

<u>then</u> $E(X_t) = \lambda \int g(t)\, dt$ and $r_X(\tau) = \lambda \int g(u) g(u-\tau)\, du$

Filtering

Impulse response $h(u)$

$$Y_t = \begin{cases} \int_{\mathbb{R}} h(u) X_{t-u}\, du & \text{in continuous time} \\ \sum_{u \in \mathbb{Z}} h(u) X_{t-u} & \text{in discrete time} \end{cases}$$

Relation to cvf

$$r_Y(\tau) = \begin{cases} \iint_{\mathbb{R}^2} h(u)h(v)r_X(\tau+u-v)\,du\,dv & \text{in continuous time} \\ \sum\sum_{(u,v)\in\mathbb{Z}^2} h(u)h(v)r_X(\tau+u-v) & \text{in discrete time} \end{cases}$$

Relation to transfer function

$$H(f) = \begin{cases} \int_{\mathbb{R}} h(u)e^{-i2\pi fu}du & \text{in continuous time} \\ \sum_{u\in\mathbb{Z}} h(u)e^{-i2\pi fu} & \text{in discrete time} \end{cases}$$

$$h(u) = \begin{cases} \int_{\mathbb{R}} H(f)e^{i2\pi fu}df & \text{in continuous time} \\ \sum_{u\in\mathbb{Z}} H(f)e^{i2\pi fu} & \text{in discrete time} \end{cases}$$

Relation between spectral density and transfer function

$$R_Y(f) = |H(f)|^2 R_X(f)$$

Expectation and covariance of random integrals

$$E\Big(\int g(s)X_s\,ds\Big) = \int_{\mathbb{R}} g(s)E(X_s)\,ds$$

$$C(\int g(s)X_s\,ds\ ,\ \int h(t)Y_t\,dt\Big) = \int\int_{\mathbb{R}^2} g(s)h(t)C(X_s,Y_t)\,ds\,dt$$

Def White noise in discrete time is an independent sequence $\{\epsilon_t\}_{t=-\infty}^{\infty}$ with $E(\epsilon_t)=0$ and $V(\epsilon_t)=\sigma_\epsilon^2$, which implies that $R_\epsilon(f)=\sigma_\epsilon^2$ for $f\in(-\frac{1}{2},\frac{1}{2}]$.

Def An $AR(p)$ process, $\{X_t : t\in\mathbb{Z}\}$, is definied as a weakly stationary process satisfying $E(X_t)=0$ and $X_t + a_1X_{t-1} + a_2X_{t-2} + \ldots + a_pX_{t-p} = \epsilon_t$ where $\{\epsilon_t\}$ is white noise such that $\epsilon_t \perp X_s$ whenever $t>s$.

Thm For an $AR(p)$ process we have

(Yule Walker equations)

$$r_X(\tau) + a_1r_X(\tau-1) + \ldots + a_pr_X(\tau-p) = \begin{cases} \sigma_\epsilon^2 & \text{for } \tau=0 \\ 0 & \text{for } \tau=1,2,\ldots \end{cases}$$

(Spectral density function)

$$R_X(f) = \sigma_\epsilon^2 \Big/ |\textstyle\sum_{k=0}^{p} a_ke^{-i2\pi fk}|^2 \quad f\in(-\frac{1}{2},\frac{1}{2}]$$

Def An $MA(q)$ process, $\{X_t : t\in\mathbb{Z}\}$, is definied as a weakly stationary process satsisfying $X_t = c_0\epsilon_t + c_1\epsilon_{t-1} + c_2\epsilon_{t-2} + \ldots + c_q\epsilon_{t-q}$ where $\{\epsilon_t\}$ is white noise.

Thm For an $MA(q)$ process we have

(Covariance function)

$$r_X(\tau) = \begin{cases} \sigma_\epsilon^2 \sum_{j-k=\tau} c_jc_k & \text{for } |\tau|\le q \\ 0 & \text{for } |\tau|>q \end{cases}$$

(Spectral density function)

$$R_X(f) = \sigma_\epsilon^2 |\textstyle\sum_{k=0}^{q} c_ke^{-i2\pi fk}|^2 \quad f\in(-\frac{1}{2},\frac{1}{2}]$$

Def The cross covariance of $\{X_t\}$ and $\{Y_t\}$ is denoted by $r_{X,Y}(\tau)$ and defined by $r_{X,Y}(\tau) = C(X_t, Y_{t+\tau}) = \int_{\mathbb{R}} e^{i2\pi f\tau} R_{X,Y}(f)\, df$

Derivative process

Thm If $r''_X(\tau)$ exists then $\{X'_t\}$ exists and then
$r_{X'}(\tau) = -r''_X(\tau) \quad R_{X'}(f) = (2\pi f)^2 R_X(f) \quad r_{X,X'}(\tau) = r'_X(\tau)$

Inference

Thm (Estimation of expected value)

If $\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$

then $V(\hat{m}_n) = \frac{1}{n^2}\sum_{k=-n+1}^{n-1}(n-|k|)r_X(k)$

$nV(\hat{m}_n) \approx \sum_{k=-\infty}^{\infty} r_X(k)$ for large n

Thm (Estimation of cvf and periodogram)

If m is known

then an unbiased estimator of $r_X(\tau)$ is

$\hat{r}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m)(X_{t+\tau} - m) \quad$ for $\tau \geq 0$

If $z_n(f) = \sum_{t=0}^{n-1} X_t e^{-i2\pi ft}$

then $\hat{R}_{per}(f) = \frac{1}{n}|z_n(f)|^2$

$E(\hat{R}_{per}(f)) = \sum_{|\tau|<n} e^{-i2\pi f\tau}(1 - \frac{|\tau|}{n})r_X(\tau)$

$$V(\hat{R}_{per}(f)) \approx \begin{cases} R(f)^2\left(1 + \left(\frac{\sin 2\pi nf}{n\sin 2\pi f}\right)^2\right) & \text{for } 0 < |f| < \frac{1}{2} \\ R(f)^2 & \text{for } f = 0, \pm\frac{1}{2} \end{cases}$$

$C(\hat{R}_{per}(f)\,,\, \hat{R}_{per}(f')) \to 0$ for $f \neq f'$ as $n \to \infty$

$2\hat{R}_{per}(f)/R(f)$ is approxiamtely $\chi^2(2)$ distributed for $0 < f < \frac{1}{2}$

Thm (Time and frequency window)

$k_n(\tau) = k(\tau/L_n)$

$K(f) = \int_{-1}^{1} e^{i2\pi f\tau} k(\tau)\, d\tau$

$K_n(f) = \sum_{\tau=-L_n}^{L_n} e^{i2\pi f\tau} k_n(\tau) \approx L_n K(fL_n)$

(Smoothed periodogram)

$\hat{R}(f) = \sum_{\tau=-L_n}^{L_n} e^{i2\pi f\tau} k_n(\tau)\hat{r}_n(\tau)$

$E(\hat{R}(f)) \approx L_n \int_{-1/2}^{1/2} K((x-f)L_n)\, dx$ ($\approx R(f)$ if L_n is large)

$V(\hat{R}(f)) = \frac{L_n}{n} R(f)^2 \int_{-1}^{1} k(x)^2 dx$

Point estimation If $E(X) = \mu$, $V(X) = \sigma^2$ and $X_1, \ldots, X_n$ is a sample of X then exampels of point estimators of μ and σ^2 are:

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\hat{\sigma}^2 = S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$$

if μ is known

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\,\bar{X}^2\right)$$

if μ is unknown.

Def A point estimator, θ^*, of a parameter θ is unbiased if $E(\theta^*) = \theta$. If θ_1^* and θ_2^* are unbiased estimators of θ, then θ_1^* is better/more efficient than θ_2^* om $V(\theta_1^*) < V(\theta_2^*)$.

Distributions, expected values and variances

	X	$p(x), f(x)$	$E(X)$	$V(X)$
Discrete distributions	Unif(N)	$1/N$, $x = 1, 2, \ldots, N$	$(N+1)/2$	$(N^2-1)/12$
	Bin(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
	Poi(λ)	$e^{-\lambda}\lambda^x/x!$, $x = 0, 1, 2, \ldots$	λ	λ
	Geo(π)	$(1-\pi)^{x-1}\pi$, $x = 1, 2, 3, \ldots$	$1/\pi$	$(1-\pi)/\pi^2$
Cont. distributions	R(a, b)	$1/(b-a)$, $a \le x \le b$	$(a+b)/2$	$(a-b)^2/12$
	Exp(λ)	$\lambda e^{-\lambda x}$, $x > 0$	$1/\lambda$	$1/\lambda^2$
	N(μ, σ)	$(\sigma\sqrt{2\pi})^{-1} e^{-(x-\mu)^2/(2\sigma^2)}$, $x \in \mathbb{R}$	μ	σ^2

Normal distribution values

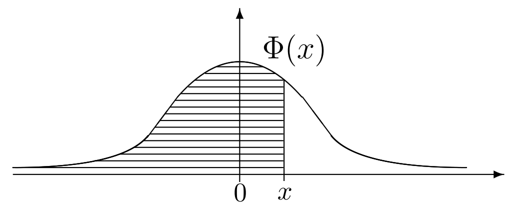

Table over values of $\Phi(x) = P(X \leq x)$ where $X \in N(0,1)$. For $x < 0$, use the relation $\Phi(x) = 1 - \Phi(-x)$.

x	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

x	+0.0	+0.1	+0.2	+0.3	+0.4	+0.5	+0.6	+0.7	+0.8	+0.9
3	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Percentiles:

Some values of λ_α such that $P(X > \lambda_\alpha) = \alpha$ where $X \in N(0,1)$

α	λ_α	α	λ_α
0.1	1.281552	0.005	2.575829
0.05	1.644854	0.001	3.090232
0.025	1.959964	0.0005	3.290527
0.01	2.326348	0.0001	3.719016

t percentiles

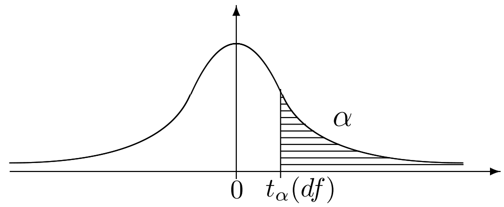

Table over values of $t_\alpha(df)$.

df \ α	0.25	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	1.0000	3.0777	6.3138	12.7062	15.8945	31.8205	63.6567	318.3088
2	0.8165	1.8856	2.9200	4.3027	4.8487	6.9646	9.9248	22.3271
3	0.7649	1.6377	2.3534	3.1824	3.4819	4.5407	5.8409	10.2145
4	0.7407	1.5332	2.1318	2.7764	2.9986	3.7470	4.6041	7.1732
5	0.7267	1.4759	2.0150	2.5706	2.7565	3.3649	4.0322	5.8934
6	0.7176	1.4398	1.9432	2.4469	2.6122	3.1427	3.7074	5.2076
7	0.7111	1.4149	1.8946	2.3646	2.5168	2.9980	3.4995	4.7853
8	0.7064	1.3968	1.8595	2.3060	2.4490	2.8965	3.3554	4.5008
9	0.7027	1.3830	1.8331	2.2622	2.3984	2.8214	3.2498	4.2968
10	0.6998	1.3722	1.8125	2.2281	2.3593	2.7638	3.1693	4.1437
12	0.6955	1.3562	1.7823	2.1788	2.3027	2.6810	3.0545	3.9296
14	0.6924	1.3450	1.7613	2.1448	2.2638	2.6245	2.9768	3.7874
17	0.6892	1.3334	1.7396	2.1098	2.2238	2.5669	2.8982	3.6458
20	0.6870	1.3253	1.7247	2.0860	2.1967	2.5280	2.8453	3.5518
25	0.6844	1.3163	1.7081	2.0595	2.1666	2.4851	2.7874	3.4502
30	0.6828	1.3104	1.6973	2.0423	2.1470	2.4573	2.7500	3.3852
50	0.6794	1.2987	1.6759	2.0086	2.1087	2.4033	2.6778	3.2614
100	0.6770	1.2901	1.6602	1.9840	2.0809	2.3642	2.6259	3.1737

χ^2 percentiles

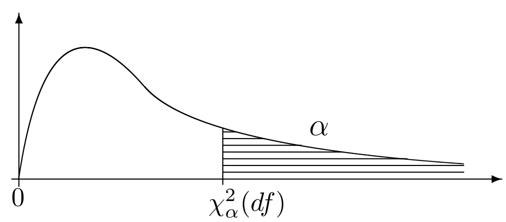

Table over values of $\chi^2_\alpha(df)$.

df \ α	0.999	0.995	0.99	0.95	0.05	0.01	0.005	0.001
1	0.0000	0.0000	0.0002	0.0039	3.8415	6.6349	7.8794	10.8276
2	0.0020	0.0100	0.0201	0.1026	5.9915	9.2103	10.5966	13.8155
3	0.0243	0.0717	0.1148	0.3518	7.8147	11.3449	12.8382	16.2662
4	0.0908	0.2070	0.2971	0.7107	9.4877	13.2767	14.8603	18.4668
5	0.2102	0.4117	0.5543	1.1455	11.0705	15.0863	16.7496	20.5150
6	0.3811	0.6757	0.8721	1.6354	12.5916	16.8119	18.5476	22.4577
7	0.5985	0.9893	1.2390	2.1673	14.0671	18.4753	20.2777	24.3219
8	0.8571	1.3444	1.6465	2.7326	15.5073	20.0902	21.9550	26.1245
9	1.1519	1.7349	2.0879	3.3251	16.9190	21.6660	23.5894	27.8772
10	1.4787	2.1559	2.5582	3.9403	18.3070	23.2093	25.1882	29.5883
12	2.2142	3.0738	3.5706	5.2260	21.0261	26.2170	28.2995	32.9095
14	3.0407	4.0747	4.6604	6.5706	23.6848	29.1412	31.3193	36.1233
17	4.4161	5.6972	6.4078	8.6718	27.5871	33.4087	35.7185	40.7902
20	5.9210	7.4338	8.2604	10.8508	31.4104	37.5662	39.9968	45.3147
25	8.6493	10.5197	11.5240	14.6114	37.6525	44.3141	46.9279	52.6197
30	11.5880	13.7867	14.9535	18.4927	43.7730	50.8922	53.6720	59.7031
50	24.6739	27.9907	29.7067	34.7643	67.5048	76.1539	79.4900	86.6608
100	61.9179	67.3276	70.0649	77.9295	124.342	135.807	140.169	149.449

Values of the Poisson distribuion

Table over values of $F(x) = P(X \leq x)$ where $X \in Po(\lambda)$.

λ	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0.5	0.607	0.910	0.986	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	0.368	0.736	0.920	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
3	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000	1.000	1.000	1.000
4	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000	1.000
5	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.986	0.995	0.998	0.999
6	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.957	0.980	0.991	0.996

Values of the Binomial distribution

Table over values of $P(x) = P(X \leq x)$ where $X \in Bin(n, p)$.
For $p > 0.5$, use the relation $P(X \leq x) = P(Y \geq n - x)$ where $Y \in Bin(n, 1-p)$.

n	p	0	1	2	3	4	5	6	7	8	9	10
3	0.1	0.729	0.972	0.999	1.000	–	–	–	–	–	–	–
	0.2	0.512	0.896	0.992	1.000	–	–	–	–	–	–	–
	0.3	0.343	0.784	0.973	1.000	–	–	–	–	–	–	–
	0.4	0.216	0.648	0.936	1.000	–	–	–	–	–	–	–
	0.5	0.125	0.500	0.875	1.000	–	–	–	–	–	–	–
4	0.1	0.656	0.948	0.996	1.000	1.000	–	–	–	–	–	–
	0.2	0.410	0.819	0.973	0.998	1.000	–	–	–	–	–	–
	0.3	0.240	0.652	0.916	0.992	1.000	–	–	–	–	–	–
	0.4	0.130	0.475	0.821	0.974	1.000	–	–	–	–	–	–
	0.5	0.062	0.312	0.688	0.938	1.000	–	–	–	–	–	–
5	0.1	0.590	0.919	0.991	1.000	1.000	1.000	–	–	–	–	–
	0.2	0.328	0.737	0.942	0.993	1.000	1.000	–	–	–	–	–
	0.3	0.168	0.528	0.837	0.969	0.998	1.000	–	–	–	–	–
	0.4	0.078	0.337	0.683	0.913	0.990	1.000	–	–	–	–	–
	0.5	0.031	0.188	0.500	0.812	0.969	1.000	–	–	–	–	–
6	0.1	0.531	0.886	0.984	0.999	1.000	1.000	1.000	–	–	–	–
	0.2	0.262	0.655	0.901	0.983	0.998	1.000	1.000	–	–	–	–
	0.3	0.118	0.420	0.744	0.930	0.989	0.999	1.000	–	–	–	–
	0.4	0.047	0.233	0.544	0.821	0.959	0.996	1.000	–	–	–	–
	0.5	0.016	0.109	0.344	0.656	0.891	0.984	1.000	–	–	–	–
7	0.1	0.478	0.850	0.974	0.997	1.000	1.000	1.000	1.000	–	–	–
	0.2	0.210	0.577	0.852	0.967	0.995	1.000	1.000	1.000	–	–	–
	0.3	0.082	0.329	0.647	0.874	0.971	0.996	1.000	1.000	–	–	–
	0.4	0.028	0.159	0.420	0.710	0.904	0.981	0.998	1.000	–	–	–
	0.5	0.008	0.062	0.227	0.500	0.773	0.938	0.992	1.000	–	–	–
8	0.1	0.430	0.813	0.962	0.995	1.000	1.000	1.000	1.000	1.000	–	–
	0.2	0.168	0.503	0.797	0.944	0.990	0.999	1.000	1.000	1.000	–	–
	0.3	0.058	0.255	0.552	0.806	0.942	0.989	0.999	1.000	1.000	–	–
	0.4	0.017	0.106	0.315	0.594	0.826	0.950	0.991	0.999	1.000	–	–
	0.5	0.004	0.035	0.145	0.363	0.637	0.855	0.965	0.996	1.000	–	–
9	0.1	0.387	0.775	0.947	0.992	0.999	1.000	1.000	1.000	1.000	1.000	–
	0.2	0.134	0.436	0.738	0.914	0.980	0.997	1.000	1.000	1.000	1.000	–
	0.3	0.040	0.196	0.463	0.730	0.901	0.975	0.996	1.000	1.000	1.000	–
	0.4	0.010	0.071	0.232	0.483	0.733	0.901	0.975	0.996	1.000	1.000	–
	0.5	0.002	0.020	0.090	0.254	0.500	0.746	0.910	0.980	0.998	1.000	–
10	0.1	0.349	0.736	0.930	0.987	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	0.2	0.107	0.376	0.678	0.879	0.967	0.994	0.999	1.000	1.000	1.000	1.000
	0.3	0.028	0.149	0.383	0.650	0.850	0.953	0.989	0.998	1.000	1.000	1.000
	0.4	0.006	0.046	0.167	0.382	0.633	0.834	0.945	0.988	0.998	1.000	1.000
	0.5	0.001	0.011	0.055	0.172	0.377	0.623	0.828	0.945	0.989	0.999	1.000

Trigonometrics

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha}) \qquad \tan\alpha = \frac{e^{i\alpha} - e^{i\alpha}}{i(e^{i\alpha}+e^{-i\alpha})}$

Some special sums and series

Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

If $|a| < 1$ then $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $\sum_{k=1}^{\infty} k a^k = \frac{a}{(1-a)^2}$ and $\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1-a)$

Taylor series: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, $\quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

Fourier transform

The <u>Fourier transform</u> of g is denoted by $G = \mathcal{F}g$ and defined by $G(f) = \int e^{-i2\pi f\tau} g(\tau)\, d\tau$.

The <u>inverse Fourier transform</u> of G is denoted by $g = \mathcal{F}^{-1}G$ and defined by $g(\tau) = \int e^{i2\pi f\tau} G(f)\, df$.

Tables of Fourier transforms

Some special cases

$g(\tau)$	$G(f)$
$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}\, e^{-2\pi\alpha\lvert f\rvert}$
$\lvert\tau\rvert e^{-\alpha\lvert\tau\rvert}$	$2\,\frac{\alpha^2-(2\pi f)^2}{(\alpha^2+(2\pi f)^2)^2}$
$\lvert\tau\rvert^k e^{-\alpha\lvert\tau\rvert}$	$\frac{k!\left((\alpha+i2\pi f)^{k+1}+(\alpha-i2\pi f)^{k+1}\right)}{(\alpha^2+(2\pi f)^2)^{k+1}}$
$e^{-\alpha\tau^2}$	$\sqrt{\frac{\pi}{\alpha}}\, e^{-(\pi f)^2/\alpha}$
$e^{-\alpha\lvert\tau\rvert}\cos(2\pi f_0\tau)$	$\frac{\alpha}{\alpha^2+4\pi^2(f_0-f)^2} + \frac{\alpha}{\alpha^2+4\pi^2(f_0+f)^2}$
$e^{-\alpha\lvert\tau\rvert}\sin(2\pi f_0\tau)$	$\frac{2\pi(f_0-f)}{\alpha^2+4\pi^2(f_0-f)^2} + \frac{2\pi(f_0-f)}{\alpha^2+4\pi^2(f_0+f)^2}$
$\begin{cases} \alpha & \text{if } \tau = 0 \\ \frac{\sin(2\pi\alpha\tau)}{2\pi\tau} & \text{if } \tau \neq 0 \end{cases}$	$\begin{cases} 1/2 & \text{if } \lvert f\rvert \leq \alpha \\ 0 & \text{if } \lvert f\rvert > \alpha \end{cases}$
$\begin{cases} 1-\alpha\lvert\tau\rvert & \text{if } \lvert\tau\rvert \leq \frac{1}{\alpha} \\ 0 & \text{if } \lvert\tau\rvert > \frac{1}{\alpha} \end{cases}$	$\begin{cases} \frac{1}{\alpha} & \text{if } f = 0 \\ \frac{2\alpha\left(1-\cos\left(\frac{2\pi f}{\alpha}\right)\right)}{(2\pi f)^2} & \text{if } f \neq 0 \end{cases}$

Some general rules

$g(\tau)$	$G(f)$
$\delta_c(\tau)$	$e^{-i2\pi fc}$
$e^{i2\pi c\tau}$	$\delta_c(f)$
$H(\tau)$	$h(-f)$
$H(-\tau)$	$h(f)$
$h(\alpha\tau)$	$\frac{1}{\alpha}H\left(\frac{f}{\alpha}\right)$
$\frac{1}{\alpha}h\left(\frac{\tau}{\alpha}\right)$	$H(\alpha f)$
$h(\tau-\tau_0)$	$H(f)e^{-i2\pi f\tau_0}$
$h(\tau)e^{i2\pi f_0\tau}$	$H(f-f_0)$

where $H = \mathcal{F}h$