Solutions to Exam for Random Processes, 7.5 ECTS

January 13, 2012, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume that $\{X_t\}$ is a weakly stationary process which is filtered through a linear filter with frequency function H resulting in the output signal $\{Y_t\}$. Prove that $R_Y(f) = |H(f)|^2 R_X(f)$.(3p)

   Solution: (See p 84, *Random Processes*.) □

2. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process with expectation function $m(t) = 1$ and covariance function $r(\tau) = \pi e^{-\pi|\tau|}\cos(\pi\tau)$. Calculate

   (a) $P(X_t + X_{t+1} < 1)$. (3p)

   (b) the spectral density of $\{X_t\}$. (3p)

   Solution:

   (a) $\{X_t\}$ Gaussian ⇒ $X_t + X_{t+1} \in N(\mu, \sigma^2)$ where $\mu = E(X_t + X_{t+1}) = 1 + 1 = 2$ and $\sigma^2 = C(X_t + X_{t+1}, X_t + X_{t+1}) = 2r(0) + 2r(1) = 2e^0 + 2e^{-\pi} = 2(1 - e^{-\pi}) = 1.9136$ and thus $P(X_t + X_{t+1} < 1) = \Phi(\frac{1-2}{\sqrt{1.936}}) = 1 - \Phi(0.72) = 0.2358$.

   (b) Since $r(\tau) = \pi e^{-\pi|\tau|}\cos(\pi\tau)$ we have according to the Fourier transform table wiht $\alpha = \pi$ and $f_0 = \frac{1}{2}$ that $R(f) = \pi\left(\frac{\pi}{\pi^2+4\pi^2(\frac{1}{2}-f)^2} + \frac{\pi}{\pi^2+4\pi^2(\frac{1}{2}+f)^2}\right) = \frac{1}{1+4\cdot\frac{1}{4}(1-2f)^2} + \frac{1}{1+4\cdot\frac{1}{4}(1+2f)^2} = \frac{2+4f+4f^2+2-4f+4f^2}{(2-4f+4f^2)(2+4f+4f^2)} = \frac{1+2f}{1+4f^2}$. □

3. Assume that the weakly stationary process $\{Y_t : t \in \mathbb{Z}\}$ is defined by the recursive relationship $Y_t = e_t + \frac{1}{2}e_{t-1} + \frac{1}{3}Y_{t-1}$ where $\{e_t : t \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables with $E(e_t) = 0$ and $V(e_t) = 2$.

   (a) What is a random process like $\{e_t\}$ called? (1p)

   (b) What is a random process like $\{Y_t\}$ called? (2p)

   (c) Calculate $C(Y_t, Y_{t-1})$. (4p)

   Solution:

   (a) White noise.

   (b) $ARMA(1,1)$.

   (c) $r(1) = C(e_t + \frac{1}{2}e_{t-1} + \frac{1}{3}Y_{t-1},\, e_{t+1} + \frac{1}{2}e_t + \frac{1}{3}Y_t) = 0 + \frac{1}{2}V(e_t) + \frac{1}{3}C(e_t, Y_t) + 0 + 0 + \frac{1}{6}C(e_{t-1}, Y_t) + 0 + 0 + \frac{1}{9}r(1) = \frac{1}{2}\cdot 2 + \frac{1}{3}C(e_t, e_t + \frac{1}{3}C(e_t, e_t + \frac{1}{2}e_{t-1} + \frac{1}{3}Y_{t-1}) + \frac{1}{6}C(e_{t-1}, e_t + \frac{1}{3}C(e_t, e_t + \frac{1}{2}e_{t-1} + \frac{1}{3}Y_{t-1}) + \frac{1}{9}r(1) = 1 + \frac{1}{3}\cdot 2 + \frac{1}{6}(0 + \frac{1}{2}\cdot 2 + \frac{1}{3}C(e_{t-1}, Y_{t-1})) + \frac{1}{9}r(1) = \frac{5}{3} + \frac{1}{6} + \frac{1}{18}\cdot 2 + \frac{1}{9}r(1) = (1 - \frac{1}{9})r(1) = \frac{35}{18} = r(1) = \frac{35}{16} = 2.1875$. □

4. Let $\{X_t : t \in \mathbb{R}\}$ be shot noise with intensity λ and and impulse function

$$g(t) = \begin{cases} (1-t)^2 & \text{when } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the expectation function of $\{X_t\}$. (3p)

(b) What values of λ makes the mean effect bounded by 1, i.e. what λ values makes $E(X_t^2) < 1$? (3p)

Solution:

(a) According to the Campbells formulae we get $E(X_t) = \lambda \int g(u)\, du = \lambda \int_0^1 (1-u)^2 du = \lambda(\int_0^1 du - 2\int_0^1 u\, du + \int_0^1 u^2 du) = \lambda(1 - 2[\frac{u^2}{2}]_0^1 + [\frac{u^3}{3}]_0^1 = \lambda(1 - 1 + \frac{1}{3}) = \frac{\lambda}{3}$.

(b) $E(X_t^2) - E(X_t)^2 = V(X_t) \;=\; E(X_t^2) = V(X_t) - E(X_t)^2$. $V(X_t) = \lambda \int g(u)g(u-0)\, du = \lambda \int_0^1 (1-u)^4 du = \lambda(\int_0^1 du - 4\int_0^1 u\, du + 6\int_0^1 u^2 du - 4\int_0^1 u^3 du + \int_0^1 u^4 du) = \lambda(1-2+2-1+\frac{1}{5}) = \frac{\lambda}{5} \;=\; E(X_t^2) = \frac{\lambda}{5} - (\frac{\lambda}{3})^2 = 1$, i.e. $\lambda^2 + \frac{9}{5}\lambda - 9 = 0 \;=\; \lambda = \frac{9}{10} \pm \sqrt{\frac{81}{100} + 9} \;=\; \lambda_1 = 4.0321$, $\lambda_2 = -2.2321$. Thus $\frac{\lambda}{5} + (\frac{\lambda}{3})^2 < 1$ means that $-2.23 < \lambda < 4.03$. But since λ is the intensity parameter of a Poisson process (driving the shot noise process), it has to be positive, and therefore the mean effect $E(X_t^2)$ is bounded by 1 for all values of $\lambda < 4.03$.□

5. Calculate $V(\int_0^1 X_t dt)$ if $\{X_t : t \in \mathbb{R}\}$ is a weakly stationary process with covariance function $r(\tau) = 1 - \tau^2$ when $|\tau| < 1$ and 0 otherwise. (4p)

Solution: $V\left(\int\limits_0^1 X_t\, dt\right) = C\left(\int\limits_0^1 X_s\, ds, \int\limits_0^1 X_t\, dt\right) = \int\limits_0^1\int\limits_0^1 C(X_s, X_t)\, ds\, dt = \int\limits_0^1\int\limits_0^1 (1-(s-t)^2)\, ds\, dt = \int\limits_0^1 \left[s - \frac{s^3}{3} + s^2 t - t^2 s\right]_0^1 dt = \left[\frac{2}{3}t + \frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 = \frac{5}{6}$. □

6. Let $f(x) = \sum_{j=1}^{\infty} \frac{\delta_{q_j}(x)}{j!}$ where $\{q_j : j \in \mathbb{Z}^+\}$ are the positive rational numbers ($\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$). Determine the value of $\int_{\mathbb{R}} f(x)dx$. (4p)

Solution: Since $\sum\limits_{j=0}^{\infty} \frac{1}{j!} = e$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{j=0}^{\infty} \frac{\delta_{q_j}(x)}{j!}\, dx &= \sum_{j=1}^{\infty} \frac{1}{j!} \cdot \underbrace{\int_{\mathbb{R}} \delta_{q_j}(x)\, dx}_{=1 \text{ since all } q_j \in \mathbb{R}} \\
&= \sum_{j=1}^{\infty} \frac{1}{j!} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} - 1 \\
&= e - 1
\end{aligned}$$

□