

# Multivariable calculus, 2007-10-26.

Per-Sverre Svendsen, Tel.035 - 167 615/0709 - 398 526.

Minimum requirements: Grade 5: 27p, 4: 21p, 3: 15p.

You are allowed to use one unmarked, ordinary (non-mathematical) dictionary of your choice. No additional material is allowed.

1. Find an equation of the plane tangent to the surface  $e^{2x+yz} - z = 0$  at the point  $(0, 0, 1)$ . (2p)

2. Find the following limit or show that it does not exist  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{3x^2 + 3y^2 + x^3 - y^3}$ . (2p)

3. Find a point  $P = (a, b)$  such that the directional derivatives of  $f(x, y) = x^2 + yx$  at  $(a, b)$  are  $f'_{\mathbf{u}}(a, b) = 8$  and  $f'_{\mathbf{v}}(a, b) = 2$  in the directions  $\mathbf{u} = (1, 0)$  and  $\mathbf{v} = (3, -4)$ , respectively. (2p)

4. The equation  $\sqrt{x} + y^2 + z + \sin(xz) = 2$  implicitly defines a function  $z = f(x, y)$  for which  $f(1, 1) = 0$ . Calculate  $f_x(1, 1)$  and  $f_y(1, 1)$ . (3p)

5. Calculate  $\iint_D \frac{2x + y}{(x - 2y)^2 + 1} dx dy$ ,  $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq 2x + y \leq 2, 0 \leq x - 2y \leq 1\}$ . (3p)

6. Calculate  $\iint_S x dS$ , where the surface  $S$  is given by  
 $S: \mathbf{r} = (x, y, z) = (x, y, \sqrt{2x^2 + 2y^2}), x \leq 1, -x \leq y \leq x$ . (3p)

7. Find the (absolute) minimum and maximum values of  $f(x, y) = 2x^2 + y^2 - 2xy - 2y$  on the set  $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y \leq 4, x \geq 0\}$ . (5p)

8. Find the (absolute) minimum and maximum values of  $f(x, y, z) = z$  subject to the constraints  $x^2 + 2y^2 = 6$  and  $x + y + z = 1$ . (5p)

9. Calculate  $\iiint_K z^2 dx dy dz$ ,  $K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 2, x^2 + y^2 \geq 1\}$ . (5p)

# Reference formulas and equations in Multivariable Calculus

## Trigonometry and Logarithms

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\pm \cot x + \cot y}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\ln x + \ln y = \ln xy$$

$$\ln x^a = a \ln x$$

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$2 \sin x \sin y = \cos(x-y) - \cos(x+y)$$

$$2 \cos x \cos y = \cos(x-y) + \cos(x+y)$$

$$2 \sin x \cos y = \sin(x-y) + \sin(x+y)$$

$$\ln x - \ln y = \ln \frac{x}{y}$$

$$(x, y > 0)$$

## Standard limits

$$\lim_{x \rightarrow 0^+} x^\alpha \log_a x = 0 \quad (a > 1, \alpha > 0) \quad \lim_{x \rightarrow \infty} \frac{a^x}{x^\alpha} = \infty \quad (a > 1)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow \infty} \frac{x^\alpha}{\log_a x} = \infty \quad (a > 1, \alpha > 0)$$

$$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$$

## Basic derivatives

$f(x)$	$f'(x)$
$x^a$	$ax^{a-1}$
$a^x$	$a^x \ln a$
$\ln  x $	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\ln  x + \sqrt{x^2 + \alpha} $	$\frac{1}{\sqrt{x^2 + \alpha}}$
$\frac{1}{2}x\sqrt{x^2 + \alpha} + \frac{\alpha}{2} \ln  x + \sqrt{x^2 + \alpha} $	$\sqrt{x^2 + \alpha}$

## Taylor Series

**Taylor's formula for a function**  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n$$

**Table of particular expansions** ( $a = 0$ ,  $h \rightarrow x$ )

1.  $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$  ( $-1 < x < 1$ )
2.  $(x+1)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2 \cdot 3}x^3 + \dots$  ( $-1 < x < 1$ )
3.  $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$
4.  $\sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$
5.  $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$
6.  $\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$  ( $-1 < x \leq 1$ )
7.  $\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} x^{2k-1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$  ( $-1 \leq x \leq 1$ )

**Taylor's formula for a function**  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(a+h, b+k) = f(a, b) + hf_x(a, b) + kf_y(a, b) + \frac{1}{2}(h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)) + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x, y)_{(a, b)}$$

## Tangent plane

**Function**  $z = f(x, y)$

Equation of tangent plane through the point  $(a, b, f(a, b))$

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

**Level surface**  $F(x, y, z) = C$

Equation of tangent plane through the point  $(a, b, c)$

$$F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$$

## Directional derivative

The directional derivative of a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  at the point  $(a, b, c)$  and direction  $\mathbf{u}$  ( $|\mathbf{u}| = 1$ )

$$D_{\mathbf{u}}f(a, b, c) = f'_{\mathbf{u}}(a, b, c) = \mathbf{u} \cdot \nabla f(a, b, c) = \mathbf{u} \cdot (f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)).$$

## Double Integrals

### General substitution

Assume a one-to-one mapping between a region  $D$  in the  $xy$ -plane and a region  $D_{uv}$  in the  $uv$ -plane

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} \Leftrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

$$\text{Then } \iint_D f(x, y) \, dx dy = \iint_{D_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du dv,$$

$$\text{with } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0.$$

### Polar coordinates

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \quad \iint_D f(x, y) \, dx dy = \iint_{D_{r\theta}} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

## Triple Integrals

### General substitution

As above assume a one-to-one mapping between points  $(x, y, z)$  in  $\Delta$  and  $(u, v, w)$  in  $\Delta_{uvw}$ .

$$\iiint_{\Delta} f(x, y, z) \, dx dy dz = \iiint_{\Delta_{uvw}} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du dv dw,$$

$$\text{with } \frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0$$

### Spherical coordinates

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi$$

$$\iiint_{\Delta} f(x, y, z) \, dx dy dz = \iiint_{\Delta_{\rho\phi\theta}} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

## Line Integrals

### Line integral with respect to arc length

Given a parametrized curve  $C : \mathbf{r}(t) = (x(t), y(t), z(t))$ ,  $a \leq t \leq b$  and a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ .

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt, \quad (ds = d|\mathbf{r}|)$$

## Surface Integrals

### General parametrized surface

$S : \mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ ,  $(u, v) \in D$ .

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

### Function graph $z = h(x, y)$

$S : \mathbf{r} = (x, y, z) = (x, y, h(x, y))$ ,  $(x, y) \in S_{xy}$ .

$$\iint_S f(x, y, z) dS = \iint_{S_{xy}} f(x, y, h(x, y)) \sqrt{1 + h_x^2 + h_y^2} dx dy$$