# C Tables of Fourier Representations and Properties

## C.1 Basic Discrete-Time Fourier Series Pairs

| Time Domain | Frequency Domain |
|---|---|
| $x[n] = \sum_{k=0}^{N-1} X[k]e^{jkn\Omega_o}$ <br> Period $= N$ | $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jkn\Omega_o}$ <br> $\Omega_o = \frac{2\pi}{N}$ |
| $x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & M < \lvert n\rvert \le N/2 \end{cases}$ <br> $x[n] = x[n+N]$ | $X[k] = \dfrac{\sin\left(k\frac{\Omega_o}{2}(2M+1)\right)}{N\sin\left(k\frac{\Omega_o}{2}\right)}$ |
| $x[n] = e^{jp\Omega_o n}$ | $X[k] = \begin{cases} 1, & k = p, p \pm N, p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| $x[n] = \cos(p\Omega_o n)$ | $X[k] = \begin{cases} \frac{1}{2}, & k = \pm p, \pm p \pm N, \pm p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| $x[n] = \sin(p\Omega_o n)$ | $X[k] = \begin{cases} \frac{1}{2j}, & k = p, p \pm N, p \pm 2N, \ldots \\ \frac{-1}{2j}, & k = -p, -p \pm N, -p \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| $x[n] = 1$ | $X[k] = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| $x[n] = \sum_{p=-\infty}^{\infty}\delta[n - pN]$ | $X[k] = \frac{1}{N}$ |

## C.2 Basic Fourier Series Pairs

| *Time Domain* | *Frequency Domain* |
|---|---|
| $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ <br> *Period* $= T$ | $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ <br> $\omega_o = \frac{2\pi}{T}$ |
| $x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & T_o < \lvert t\rvert \le T/2 \end{cases}$ | $X[k] = \frac{\sin(k\omega_o T_o)}{k\pi}$ |
| $x(t) = e^{jp\omega_o t}$ | $X[k] = \delta[k-p]$ |
| $x(t) = \cos(p\omega_o t)$ | $X[k] = \frac{1}{2}\delta[k-p] + \frac{1}{2}\delta[k+p]$ |
| $x(t) = \sin(p\omega_o t)$ | $X[k] = \frac{1}{2j}\delta[k-p] - \frac{1}{2j}\delta[k+p]$ |
| $x(t) = \sum_{p=-\infty}^{\infty}\delta(t - pT)$ | $X[k] = \frac{1}{T}$ |

## C.3 Basic Discrete-Time Fourier Transform Pairs

| *Time Domain* | *Frequency Domain* |
|---|---|
| $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$ | $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ |
| $x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & \text{otherwise} \end{cases}$ | $X(e^{j\Omega}) = \frac{\sin\left[\Omega\left(\frac{2M+1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$ |
| $x[n] = \alpha^n u[n], \quad \lvert\alpha\rvert < 1$ | $X(e^{j\Omega}) = \frac{1}{1 - \alpha e^{-j\Omega}}$ |
| $x[n] = \delta[n]$ | $X(e^{j\Omega}) = 1$ |
| $x[n] = u[n]$ | $X(e^{j\Omega}) = \frac{1}{1 - e^{-j\Omega}} + \pi\sum_{p=-\infty}^{\infty}\delta(\Omega - 2\pi p)$ |
| $x[n] = \frac{1}{\pi n}\sin(Wn), \quad 0 < W \le \pi$ | $X(e^{j\Omega}) = \begin{cases} 1, & \lvert\Omega\rvert \le W \\ 0, & W < \lvert\Omega\rvert \le \pi \end{cases}$ $X(e^{j\Omega})$ is $2\pi$ periodic |
| $x[n] = (n+1)\alpha^n u[n]$ | $X(e^{j\Omega}) = \frac{1}{(1 - \alpha e^{-j\Omega})^2}$ |

## C.4 Basic Fourier Transform Pairs

| *Time Domain* | *Frequency Domain* |
|---|---|
| $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ | $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ |
| $x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & \text{otherwise} \end{cases}$ | $X(j\omega) = \frac{2\sin(\omega T_o)}{\omega}$ |
| $x(t) = \frac{1}{\pi t}\sin(Wt)$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert \le W \\ 0, & \text{otherwise} \end{cases}$ |
| $x(t) = \delta(t)$ | $X(j\omega) = 1$ |
| $x(t) = 1$ | $X(j\omega) = 2\pi\delta(\omega)$ |
| $x(t) = u(t)$ | $X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$ |
| $x(t) = e^{-at}u(t), \quad \text{Re}\{a\} > 0$ | $X(j\omega) = \frac{1}{a + j\omega}$ |
| $x(t) = te^{-at}u(t), \quad \text{Re}\{a\} > 0$ | $X(j\omega) = \frac{1}{(a + j\omega)^2}$ |
| $x(t) = e^{-a\lvert t\rvert}, \quad a > 0$ | $X(j\omega) = \frac{2a}{a^2 + \omega^2}$ |
| $x(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ | $X(j\omega) = e^{-\omega^2/2}$ |

## C.5 Fourier Transform Pairs for Periodic Signals

| *Periodic Time-Domain Signal* | *Fourier Transform* |
|---|---|
| $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ | $X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o)$ |
| $x(t) = \cos(\omega_o t)$ | $X(j\omega) = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$ |
| $x(t) = \sin(\omega_o t)$ | $X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_o) - \frac{\pi}{j}\delta(\omega + \omega_o)$ |
| $x(t) = e^{j\omega_o t}$ | $X(j\omega) = 2\pi\delta(\omega - \omega_o)$ |
| $x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$ | $X(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_s}\right)$ |
| $x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & T_o < \lvert t\rvert < T/2 \end{cases}$ <br> $x(t + T) = x(t)$ | $X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{2\sin(k\omega_o T_o)}{k}\delta(\omega - k\omega_o)$ |

## C.6 *Discrete-Time Fourier Transform Pairs for Periodic Signals*

| *Periodic Time-Domain Signal* | *Discrete-Time Fourier Transform* |
|---|---|
| $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}$ | $X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$ |
| $x[n] = \cos(\Omega_1 n)$ | $X(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) + \delta(\Omega + \Omega_1 - k2\pi)$ |
| $x[n] = \sin(\Omega_1 n)$ | $X(e^{j\Omega}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) - \delta(\Omega + \Omega_1 - k2\pi)$ |
| $x[n] = e^{j\Omega_1 n}$ | $X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi)$ |
| $x[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN)$ | $X(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{k2\pi}{N}\right)$ |

## C.7 Properties of Fourier Representations

| *Property* | *Fourier Transform* $x(t) \xleftrightarrow{FT} X(j\omega)$ $y(t) \xleftrightarrow{FT} Y(j\omega)$ | *Fourier Series* $x(t) \xleftrightarrow{FS;\omega_o} X[k]$ $y(t) \xleftrightarrow{FS;\omega_o} Y[k]$ *Period* $= T$ |
|---|---|---|
| Linearity | $ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$ | $ax(t) + by(t) \xleftrightarrow{FS;\omega_o} aX[k] + bY[k]$ |
| Time shift | $x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$ | $x(t - t_o) \xleftrightarrow{FS;\omega_o} e^{-jk\omega_o t_o}X[k]$ |
| Frequency shift | $e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$ | $e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k - k_o]$ |
| Scaling | $x(at) \xleftrightarrow{FT} \frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$ | $x(at) \xleftrightarrow{FS;a\omega_o} X[k]$ |
| Differentiation in time | $\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$ | $\frac{d}{dt}x(t) \xleftrightarrow{FS;\omega_o} jk\omega_o X[k]$ |
| Differentiation in frequency | $-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$ | — |
| Integration/ Summation | $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$ | — |
| Convolution | $\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{FT} X(j\omega)Y(j\omega)$ | $\int_{0}^{T} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{FS;\omega_o} TX[k]Y[k]$ |
| Multiplication | $x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)Y(j(\omega - \nu))\,d\nu$ | $x(t)y(t) \xleftrightarrow{FS;\omega_o} \sum_{l=-\infty}^{\infty} X[l]Y[k - l]$ |
| Parseval's Theorem | $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$ | $\frac{1}{T}\int_{0}^{T} \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$ |
| Duality | $X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$ | $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ <br> $X(e^{jt}) \xleftrightarrow{FS;1} x[-k]$ |
| Symmetry | $x(t)$ real $\xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$ <br> $x(t)$ imaginary $\xleftrightarrow{FT} X^*(j\omega) = -X(-j\omega)$ <br> $x(t)$ real and even $\xleftrightarrow{FT} \operatorname{Im}\{X(j\omega)\} = 0$ <br> $x(t)$ real and odd $\xleftrightarrow{FT} \operatorname{Re}\{X(j\omega)\} = 0$ | $x(t)$ real $\xleftrightarrow{FS;\omega_o} X^*[k] = X[-k]$ <br> $x(t)$ imaginary $\xleftrightarrow{FS;\omega_o} X^*[k] = -X[-k]$ <br> $x(t)$ real and even $\xleftrightarrow{FS;\omega_o} \operatorname{Im}\{X[k]\} = 0$ <br> $x(t)$ real and odd $\xleftrightarrow{FS;\omega_o} \operatorname{Re}\{X[k]\} = 0$ |

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## C.7 *(continued)*

| Property | Discrete-Time FT $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ $y[n] \xleftrightarrow{DTFT} Y(e^{j\Omega})$ | Discrete-Time FS $x[n] \xleftrightarrow{DTFS;\Omega_o} X[k]$ $y[n] \xleftrightarrow{DTFS;\Omega_o} Y[k]$ Period $= N$ |
|---|---|---|
| Linearity | $ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\Omega}) + bY(e^{j\Omega})$ | $ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_o} aX[k] + bY[k]$ |
| Time shift | $x[n - n_o] \xleftrightarrow{DTFT} e^{-j\Omega n_o}X(e^{j\Omega})$ | $x[n - n_o] \xleftrightarrow{DTFS;\Omega_o} e^{-jk\Omega_o n_o}X[k]$ |
| Frequency shift | $e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$ | $e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\Omega_o} X[k - k_o]$ |
| Scaling | $x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \ldots$ <br> $x_z[pn] \xleftrightarrow{DTFT} X_z(e^{j\Omega/p})$ | $x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \ldots$ <br> $x_z[pn] \xleftrightarrow{DTFS;p\Omega_o} pX_z[k]$ |
| Differentiation in time | — | — |
| Differentiation in frequency | $-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X(e^{j\Omega})$ | — |
| Integration/ Summation | $\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{DTFT} \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}}$ <br> $+ \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$ | — |
| Convolution | $\sum_{l=-\infty}^{\infty} x[l]y[n - l] \xleftrightarrow{DTFT} X(e^{j\Omega})Y(e^{j\Omega})$ | $\sum_{l=0}^{N-1} x[l]y[n - l] \xleftrightarrow{DTFS;\Omega_o} NX[k]Y[k]$ |
| Multiplication | $x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma})Y(e^{j(\Omega-\Gamma)})\, d\Gamma$ | $x[n]y[n] \xleftrightarrow{DTFS;\Omega_o} \sum_{l=0}^{N-1} X[l]Y[k - l]$ |
| Parseval's Theorem | $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2\, d\Omega$ | $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$ |
| Duality | $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ <br> $X(e^{jt}) \xleftrightarrow{FS;1} x[-k]$ | $X[n] \xleftrightarrow{DTFS;\Omega_o} \frac{1}{N}x[-k]$ |
| Symmetry | $x[n]$ real $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = X(e^{-j\Omega})$ <br> $x[n]$ imaginary $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = -X(e^{-j\Omega})$ <br> $x[n]$ real and even $\xleftrightarrow{DTFT} \text{Im}\{X(e^{j\Omega})\} = 0$ <br> $x[n]$ real and odd $\xleftrightarrow{DTFT} \text{Re}\{X(e^{j\Omega})\} = 0$ | $x[n]$ real $\xleftrightarrow{DTFS;\Omega_o} X^*[k] = X[-k]$ <br> $x[n]$ imaginary $\xleftrightarrow{DTFS;\Omega_o} X^*[k] = -X[-k]$ <br> $x[n]$ real and even $\xleftrightarrow{DTFS;\Omega_o} \text{Im}\{X[k]\} = 0$ <br> $x[n]$ real and odd $\xleftrightarrow{DTFS;\Omega_o} \text{Re}\{X[k]\} = 0$ |

## C.8 Relating the Four Fourier Representations

Let

$$g(t) \xleftrightarrow{FS;\,\omega_o = 2\pi/T} G[k]$$

$$v[n] \xleftrightarrow{DTFT} V(e^{j\Omega})$$

$$w[n] \xleftrightarrow{DTFS;\,\Omega_o = 2\pi/N} W[k]$$

### C.8.1 FT Representation for a Continuous-Time Periodic Signal

$$g(t) \xleftrightarrow{FT} G(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} G[k]\delta(\omega - k\omega_o)$$

### C.8.2 DTFT Representation for a Discrete-Time Periodic-Signal

$$w[n] \xleftrightarrow{DTFT} W(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} W[k]\delta(\Omega - k\Omega_o)$$

### C.8.3 FT Representation for a Discrete-Time Nonperiodic Signal

$$v_\delta(t) = \sum_{n=-\infty}^{\infty} v[n]\delta(t - nT_s) \xleftrightarrow{FT} V_\delta(j\omega) = V(e^{j\Omega})\Big|_{\Omega=\omega T_s}$$

### C.8.4 FT Representation for a Discrete-Time Nonperiodic Signal

$$w_\delta(t) = \sum_{n=-\infty}^{\infty} w[n]\delta(t - nT_s) \xleftrightarrow{FT} W_\delta(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} W[k]\delta\left(\omega - \frac{k\Omega_o}{T_s}\right)$$

## C.9 Sampling and Aliasing Relationships

Let

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$v[n] \xleftrightarrow{DTFT} V(e^{j\Omega})$$

### C.9.1 Impulse Sampling for Continuous-Time Signals

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\frac{2\pi}{T_s}\right)\right)$$

Sampling interval $T_s$, $X_\delta(j\omega)$ is $2\pi/T_s$ periodic.

## ■ C.9.2 Sampling a Discrete-Time Signal

$$y[n] = v[qn] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} V(e^{j(\Omega - m2\pi)/q})$$

$Y(e^{j\Omega})$ is $2\pi$ periodic.

## ■ C.9.3 Sampling the DTFT in Frequency

$$w[n] = \sum_{m=-\infty}^{\infty} v[n + mN] \xleftrightarrow{DTFS;\, \Omega_o = 2\pi/N} W[k] = \frac{1}{N} V(e^{jk\Omega_o})$$

$w[n]$ is $N$ periodic.

## ■ C.9.4 Sampling the FT in Frequency

$$g(t) = \sum_{m=-\infty}^{\infty} x(t + mT) \xleftrightarrow{FS;\, \omega_o = 2\pi/T} G[k] = \frac{1}{T} X(jk\omega_o)$$

$g(t)$ is $T$ periodic.

# D Tables of Laplace Transforms and Properties

## D.1 *Basic Laplace Transforms*

| Signal $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$ | Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ | ROC |
|---|---|---|
| $u(t)$ | $\frac{1}{s}$ | $\text{Re}\{s\} > 0$ |
| $tu(t)$ | $\frac{1}{s^2}$ | $\text{Re}\{s\} > 0$ |
| $\delta(t-\tau), \quad \tau \geq 0$ | $e^{-s\tau}$ | for all $s$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\text{Re}\{s\} > -a$ |
| $te^{-at}u(t)$ | $\frac{1}{(s+a)^2}$ | $\text{Re}\{s\} > -a$ |
| $[\cos(\omega_1 t)]u(t)$ | $\frac{s}{s^2+\omega_1^2}$ | $\text{Re}\{s\} > 0$ |
| $[\sin(\omega_1 t)]u(t)$ | $\frac{\omega_1}{s^2+\omega_1^2}$ | $\text{Re}\{s\} > 0$ |
| $[e^{-at}\cos(\omega_1 t)]u(t)$ | $\frac{s+a}{(s+a)^2+\omega_1^2}$ | $\text{Re}\{s\} > -a$ |
| $[e^{-at}\sin(\omega_1 t)]u(t)$ | $\frac{\omega_1}{(s+a)^2+\omega_1^2}$ | $\text{Re}\{s\} > -a$ |

### ■ D.1.1 BILATERAL LAPLACE TRANSFORMS FOR SIGNALS THAT ARE NONZERO FOR $t < 0$

| *Signal* | *Bilateral Transform* | *ROC* |
|---|---|---|
| $\delta(t-\tau), \tau < 0$ | $e^{-s\tau}$ | for all $s$ |
| $-u(-t)$ | $\frac{1}{s}$ | $\text{Re}\{s\} < 0$ |
| $-tu(-t)$ | $\frac{1}{s^2}$ | $\text{Re}\{s\} < 0$ |
| $-e^{-at}u(-t)$ | $\frac{1}{s+a}$ | $\text{Re}\{s\} < -a$ |
| $-te^{-at}u(-t)$ | $\frac{1}{(s+a)^2}$ | $\text{Re}\{s\} < -a$ |

## D.2 *Laplace Transform Properties*

| *Signal* | *Unilateral Transform* $x(t) \xleftrightarrow{\mathcal{L}_u} X(s)$ $y(t) \xleftrightarrow{\mathcal{L}_u} Y(s)$ | *Bilateral Transform* $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ $y(t) \xleftrightarrow{\mathcal{L}} Y(s)$ | *ROC* $s \in R_x$ $s \in R_y$ |
|---|---|---|---|
| $ax(t) + by(t)$ | $aX(s) + bY(s)$ | $aX(s) + bY(s)$ | At least $R_x \cap R_y$ |
| $x(t-\tau)$ | $e^{-s\tau}X(s)$ if $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$ | $e^{-s\tau}X(s)$ | $R_x$ |
| $e^{s_o t}x(t)$ | $X(s-s_o)$ | $X(s-s_o)$ | $R_x + \text{Re}\{s_o\}$ |
| $x(at)$ | $\frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$ | $\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$ | $\frac{R_x}{\lvert a\rvert}$ |
| $x(t) * y(t)$ | $X(s)Y(s)$ if $x(t) = y(t) = 0$ for $t < 0$ | $X(s)Y(s)$ | At least $R_x \cap R_y$ |
| $-tx(t)$ | $\frac{d}{ds}X(s)$ | $\frac{d}{ds}X(s)$ | $R_x$ |
| $\frac{d}{dt}x(t)$ | $sX(s) - x(0^-)$ | $sX(s)$ | At least $R_x$ |
| $\int_{-\infty}^{t} x(\tau)\,d\tau$ | $\frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau + \frac{X(s)}{s}$ | $\frac{X(s)}{s}$ | At least $R_x \cap \{\text{Re}\{s\} > 0\}$ |

### ■ D.2.1 INITIAL-VALUE THEOREM

$$\lim_{s\to\infty} sX(s) = x(0^+)$$

This result does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is equal to or greater than the order of the denominator polynomial. In that case,

$X(s)$ would contain terms of the form $cs^k$, $k \geq 0$. Such terms correspond to the impulses and their derivatives located at time $t = 0$.

## D.2.2 Final-Value Theorem

$$\lim_{s \to 0} sX(s) = \lim_{t \to \infty} x(t)$$

This result requires that all the poles of $sX(s)$ be in the left half of the $s$-plane.

## D.2.3 Unilateral Differentiation Property, General Form

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}_u} s^n X(s) - \left.\frac{d^{n-1}}{dt^{n-1}}x(t)\right|_{t=0^-} - s\left.\frac{d^{n-2}}{dt^{n-2}}x(t)\right|_{t=0^-} - \cdots - s^{n-2}\left.\frac{d}{dt}x(t)\right|_{t=0^-} - s^{n-1}x(0^-)$$

# E Tables of z-Transforms and Properties

## E.1 Basic z-Transforms

| Signal $x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$ | Transform $X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | All $z$ |
| $u[n]$ | $\frac{1}{1 - z^{-1}}$ | $\|z\| > 1$ |
| $\alpha^n u[n]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $\|z\| > \|\alpha\|$ |
| $n\alpha^n u[n]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\|z\| > \|\alpha\|$ |
| $[\cos(\Omega_1 n)]u[n]$ | $\frac{1 - z^{-1}\cos\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$ | $\|z\| > 1$ |
| $[\sin(\Omega_1 n)]u[n]$ | $\frac{z^{-1}\sin\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$ | $\|z\| > 1$ |
| $[r^n \cos(\Omega_1 n)]u[n]$ | $\frac{1 - z^{-1}r\cos\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2 z^{-2}}$ | $\|z\| > r$ |
| $[r^n \sin(\Omega_1 n)]u[n]$ | $\frac{z^{-1}r\sin\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2 z^{-2}}$ | $\|z\| > r$ |

### E.1.1 Bilateral Transforms for Signals that Are Nonzero for $n < 0$

| *Signal* | *Bilateral Transform* | ROC |
|---|---|---|
| $u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |

## E.2 *z*-Transform Properties

| *Signal* | *Unilateral Transform* $x[n] \overset{z_u}{\longleftrightarrow} X(z)$ $y[n] \overset{z_u}{\longleftrightarrow} Y(z)$ | *Bilateral Transform* $x[n] \overset{z}{\longleftrightarrow} X(z)$ $y[n] \overset{z}{\longleftrightarrow} Y(z)$ | ROC $z \in R_x$ $z \in R_y$ |
|---|---|---|---|
| $ax[n] + by[n]$ | $aX(z) + bY(z)$ | $aX(z) + bY(z)$ | At least $R_x \cap R_y$ |
| $x[n-k]$ | See below | $z^{-k}X(z)$ | $R_x$, except possibly $\lvert z\rvert = 0, \infty$ |
| $\alpha^n x[n]$ | $X\left(\frac{z}{\alpha}\right)$ | $X\left(\frac{z}{\alpha}\right)$ | $\lvert\alpha\rvert R_x$ |
| $x[-n]$ | — | $X\left(\frac{1}{z}\right)$ | $\frac{1}{R_x}$ |
| $x[n] * y[n]$ | $X(z)Y(z)$ if $x[n] = y[n] = 0$ for $n < 0$ | $X(z)Y(z)$ | At least $R_x \cap R_y$ |
| $nx[n]$ | $-z\frac{d}{dz}X(z)$ | $-z\frac{d}{dz}X(z)$ | $R_x$, except possibly addition or deletion of $z = 0$ |

### E.2.1 Unilateral z-Transform Time-Shift Property

$$x[n-k] \overset{z_u}{\longleftrightarrow} x[-k] + x[-k+1]z^{-1} + \cdots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for } k > 0$$

$$x[n+k] \overset{z_u}{\longleftrightarrow} -x[0]z^{k} - x[1]z^{k-1} - \cdots - x[k-1]z + z^{k}X(z) \quad \text{for } k > 0$$