

Solutions Single-variable calculus , 2010-01-14.

1. We are given $f(x) = \frac{12}{1 + \sqrt{2x+3}}$. Find $f'(3)$. (2p)

Answer:

Set $u = 1 + \sqrt{2x+3}$. Using the chain-rule:

$$f'(x) = 12 \frac{d}{du} \left(\frac{1}{u} \right) \frac{d}{dx} (1 + \sqrt{2x+3}) = 12 \cdot \left(-\frac{1}{u^2} \right) \frac{2}{2\sqrt{2x+3}} = -\frac{12}{(1 + \sqrt{2x+3})^2 \sqrt{2x+3}}.$$

$$\text{Inserted: } f'(3) = -\frac{12}{(1 + \sqrt{2 \cdot 3 + 3})^2 \sqrt{2 \cdot 3 + 3}} = -\frac{1}{4}.$$

2. Find $\frac{dy}{dx}$ in terms of x and y from the equation $x^3y + y^4 = 1$. (2p)

Answer:

Taking the derivative with respect to x of the equation gives:

$$\frac{d}{dx} (x^3y + y^4) = 0 \Leftrightarrow 3x^2y + x^3 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{3x^2y}{x^3 + 4y^3}.$$

3. Find the following limit or show that it does not exist $\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - \sqrt{4-x}}{x}$. (2p)

Answer:

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - \sqrt{4-x}}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{4+x} - \sqrt{4-x})(\sqrt{4+x} + \sqrt{4-x})}{x(\sqrt{4+x} + \sqrt{4-x})} =$$

$$\lim_{x \rightarrow 0} \frac{(4+x) - (4-x)}{x(\sqrt{4+x} + \sqrt{4-x})} = \lim_{x \rightarrow 0} \frac{2}{\sqrt{4+x} + \sqrt{4-x}} = \frac{1}{2}.$$

4. Find an equation for the tangent to the curve $y = \ln(x^2 - 3)$ at the point $(2, 0)$. (3p)

Answer:

The equation for the tangent: $y - \ln(2^2 - 3) = \left(\ln(x^2 - 3) \right)'_{x=2} (x - 2) \Rightarrow$

$$y - 0 = \left(\frac{2x}{x^2 - 3} \right)_{x=2} (x - 2) \Rightarrow y = 4x - 8.$$

5. Calculate $\int_0^{\sqrt{\pi}} x \sin(x^2) dx$. (2p)

Answer:

Set $u = x^2$. This gives: $du = 2x dx$, $x = 0 \Rightarrow u = 0$, $x = \sqrt{\pi} \Rightarrow u = \pi$. Inserted:

$$\int_0^{\sqrt{\pi}} x \sin(x^2) dx = \frac{1}{2} \int_0^{\pi} \sin u du = \frac{1}{2} (-\cos u) \Big|_0^{\pi} = \frac{1}{2} (-\cos \pi + \cos 0) = 1.$$

6. Calculate $\int_0^1 \frac{x+4}{x^2+5x+6} dx$. (3p)

Answer:

$$\frac{x+4}{x^2+5x+6} = \frac{x+4}{(x+2)(x+3)} = \frac{2}{x+2} - \frac{1}{x+3}. \text{ Inserted:}$$

$$\int_0^1 \frac{x+4}{x^2+5x+6} dx = \int_1^0 \frac{2}{x+2} dx - \int_0^1 \frac{1}{x+3} dx = 2 \ln|x+2| \Big|_0^1 - \ln|x+3| \Big|_0^1 = 3 \ln 3 - 4 \ln 2.$$

7. Calculate $\int_0^2 x \ln(x+1) dx$. (3p)

Answer:

$$\int_0^2 x \ln(x+1) dx = \int_0^2 \left(\frac{1}{2}x^2\right)' \ln(x+1) dx =$$

$$\frac{1}{2}x^2 \ln(x+1) \Big|_0^2 - \int_0^2 \frac{1}{2}x^2 (\ln(x+1))' dx = 2 \ln 3 - \frac{1}{2} \int_0^2 \frac{x^2}{x+1} dx =$$

$$2 \ln 3 - \frac{1}{2} \int_0^2 \left(x-1 + \frac{1}{x+1}\right) dx = 2 \ln 3 - \frac{1}{2} \left(\frac{1}{2}x^2 - x + \ln|x+1|\right) \Big|_0^2 = \frac{3}{2} \ln 3.$$

8. Calculate the maximum area of a triangle with two sides of equal length $l = 10$. (3p)

Answer:

Let x be half the length of the third (unequal) side of the triangle and h be the height of the triangle from the the intersection of the equal sides. This means that (make a sketch):

$$x^2 + h^2 = l^2 \Rightarrow h = \sqrt{l^2 - x^2}, \text{ and the area is then:}$$

$$A = A(x) = \frac{1}{2} 2 x h = x \sqrt{l^2 - x^2} = x \sqrt{100 - x^2}.$$

The area-function is obviously continuous and defined on the closed set $[0, 10]$. It has, therefore, an absolute maximum (and minimum). The end-points $x = 0$, $x = 10$ give the minimum, whereas the maximum must then be found at an interior point where $A'(x) = 0$:

$$A'(x) = \frac{d}{dx} \left(x \sqrt{100 - x^2}\right) = \sqrt{100 - x^2} + x \frac{-2x}{2\sqrt{100 - x^2}} = \sqrt{100 - x^2} \left(1 - \frac{x^2}{100 - x^2}\right).$$

$$\text{This gives: } A'(x) = 0, \quad (0 < x < 10) \Rightarrow 1 - \frac{x^2}{100 - x^2} = 0 \Rightarrow x = 5\sqrt{2}.$$

We also note that $A'(x) > 0$ for $0 < x < 5\sqrt{2}$ and $A'(x) < 0$ for $5\sqrt{2} < x < 10$.

This means that the maximum area is $A(5\sqrt{2}) = 50$.

9. We are given the function $f(x) = \frac{x^3}{x-1}$.

Find all critical points, local maxima/minima, inflection points, solutions of the equation $f(x) = 0$, and asymptotes of the function. Use this information to sketch the graph of f . (5p)

Answer:

Zeros of the function: $f(x) = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$.

Furthermore, we note that f is differentiable for all x for which it is defined, that is $x \in D_f = (-\infty, 1) \cup (1, \infty)$. Taking the 1st derivative:

$$f'(x) = \frac{3x^2(x-1) - x^3}{(x-1)^2} = \frac{x^2(2x-3)}{(x-1)^2}.$$

In order to find any inflection points we also have to consider the 2nd derivative of f :

$$f'' = \frac{d}{dx} \left(\frac{2x^3 - 3x^2}{(x-1)^2} \right) = \frac{(6x^2 - 6x)(x-1)^2 - (2x^3 - 3x^2) 2(x-1)}{(x-1)^4} =$$

$$\frac{2x(x-1) \left(3(x-1)^2 - (2x^2 - 3x) \right)}{(x-1)^4} = \frac{2x(x^2 - 3x + 6)}{(x-1)^3}.$$

Note that $x^2 - 3x + 6 = \left(x - \frac{3}{2}\right)^2 + \frac{15}{4} > 0$ for all $x \in \mathbb{R}$. This means that the sign of f'' is entirely determined by the other two factors.

The information above can be elucidated by the following chart (with obvious abbreviations):

		<i>CP</i>				<i>CP</i>	
x		0		1		3/2	
f'	-	0	-	<i>UD</i>	-	0	+
f''	+	0	-	<i>UD</i>	+	+	+
f	\searrow	<i>IF</i>	\searrow	<i>UD</i>	\searrow	<i>min</i>	\nearrow

In conclusion: f has a local minimum for $x = \frac{3}{2}$ and an inflection point at $(0, 0)$.

Using long division we may rewrite the function as $f(x) = x^2 + x + 1 + \frac{1}{x-1}$.

The expression above tells us that the f has the vertical asymptote $x = 1$, since $f(x) \rightarrow \pm\infty$ as $x \rightarrow 1$. The function has no horizontal or oblique asymptotes.

10. Calculate the volume of the body which is obtained by rotating the curve

$$y = \frac{1}{\sqrt{1+4x^2}}, \quad 0 \leq x \leq \frac{1}{2} \quad \text{about the } x\text{-axis.} \quad (5p)$$

Answer:

The volume of the given solid of revolution is:

$$\int_0^{\frac{1}{2}} \pi \left(\frac{1}{\sqrt{1+4x^2}} \right)^2 dx = \pi \int_0^{\frac{1}{2}} \frac{1}{1+4x^2} dx = [u = 2x]$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{1+u^2} = \frac{\pi}{2} \frac{1}{1+u^2} \Big|_0^1 = \frac{\pi}{2} (\arctan 1 - \arctan 0) = \frac{\pi}{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi^2}{8}.$$