Solutions Single-variable calculus , 2010-01-14.

1. We are given $f(x) = \dfrac{12}{1+\sqrt{2x+3}}$. Find $f'(3)$. *(2p)*

   Answer:
   Set $u = 1+\sqrt{2x+3}$. Using the chain-rule:

   $$f'(x) = 12\,\frac{d}{du}\Big(\frac{1}{u}\Big)\,\frac{d}{dx}\Big(1+\sqrt{2x+3}\Big) = 12\cdot\Big(-\frac{1}{u^2}\Big)\,\frac{2}{2\sqrt{2x+3}} = -\frac{12}{(1+\sqrt{2x+3})^2\,\sqrt{2x+3}}.$$

   Inserted: $f'(3) = -\dfrac{12}{(1+\sqrt{2\cdot 3+3})^2\,\sqrt{2\cdot 3+3}} = -\dfrac{1}{4}.$

2. Find $\dfrac{dy}{dx}$ in terms of x and y from the equation $x^3y + y^4 = 1$. *(2p)*

   Answer:
   Taking the derivative with respect to x of the equation gives:

   $$\frac{d}{dx}\Big(x^3y + y^4\Big) = 0 \Leftrightarrow 3x^2y + x^3\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{3x^2y}{x^3+4y^3}.$$

3. Find the following limit or show that it does not exist $\displaystyle\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}$. *(2p)*

   Answer:

   $$\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x} = \lim_{x\to 0}\frac{(\sqrt{4+x}-\sqrt{4-x})(\sqrt{4+x}+\sqrt{4-x})}{x\,(\sqrt{4+x}+\sqrt{4-x})} =$$

   $$\lim_{x\to 0}\frac{(4+x)-(4-x)}{x\,(\sqrt{4+x}+\sqrt{4-x})} = \lim_{x\to 0}\frac{2}{\sqrt{4+x}+\sqrt{4-x}} = \frac{1}{2}.$$

4. Find an equation for the tangent to the curve $y = \ln(x^2-3)$ at the point $(2,0)$. *(3p)*

   Answer:
   The equation for the tangent: $y - \ln(2^2-3) = \Big(\ln(x^2-3)\Big)'_{x=2}(x-2) \Rightarrow$

   $$y - 0 = \Big(\frac{2x}{x^2-3}\Big)_{x=2}(x-2) \Rightarrow y = 4x - 8.$$

5. Calculate $\displaystyle\int_0^{\sqrt{\pi}} x\,\sin(x^2)\,dx$. *(2p)*

   Answer:
   Set $u = x^2$. This gives: $du = 2x\,dx$, $x = 0 \Rightarrow u = 0$, $x = \sqrt{\pi} \Rightarrow u = \pi$. Inserted:

   $$\int_0^{\sqrt{\pi}} x\,\sin(x^2)\,dx = \frac{1}{2}\int_0^{\pi}\sin u\,du = \frac{1}{2}\,(-\cos u)\Big|_0^{\pi} = \frac{1}{2}(-\cos\pi+\cos 0) = 1.$$

6. Calculate $\displaystyle\int_0^1 \frac{x+4}{x^2+5x+6}\,dx.$ *(3p)*

Answer:

$\dfrac{x+4}{x^2+5x+6} = \dfrac{x+4}{(x+2)(x+3)} = \dfrac{2}{x+2} - \dfrac{1}{x+3}$. Inserted:

$$\int_0^1 \frac{x+4}{x^2+5x+6}\,dx = \int_1^0 \frac{2}{x+2}\,dx - \int_0^1 \frac{1}{x+3}\,dx = 2\,\ln|x+2|\Big|_0^1 - \ln|x+3|\Big|_0^1 = 3\ln 3 - 4\ln 2.$$

7. Calculate $\displaystyle\int_0^2 x\ln(x+1)\,dx.$ *(3p)*

Answer:

$$\int_0^2 x\ln(x+1)\,dx = \int_0^2 \left(\frac{1}{2}x^2\right)' \ln(x+1)\,dx =$$

$$\frac{1}{2}x^2\,\ln(x+1)\Big|_0^2 - \int_0^2 \frac{1}{2}x^2\left(\ln(x+1)\right)'\,dx = 2\ln 3 - \frac{1}{2}\int_0^2 \frac{x^2}{x+1}\,dx =$$

$$2\ln 3 - \frac{1}{2}\int_0^2 \left(x-1+\frac{1}{x+1}\right)dx = 2\ln 3 - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln|x+1|\right)\Big|_0^2 = \frac{3}{2}\ln 3.$$

8. Calculate the maximum area of a triangle with *two* sides of equal length $l = 10$. *(3p)*

Answer:
Let x be *half* the length of the third (unequal) side of the triangle and h be the height of the triangle from the the intersection of the equal sides. This means that (make a sketch):

$x^2 + h^2 = l^2 \;\Rightarrow\; h = \sqrt{l^2 - x^2}$, and the area is then:

$A = A(x) = \dfrac{1}{2}\,2\,x\,h = x\,\sqrt{l^2-x^2} = x\,\sqrt{100-x^2}.$

The area-function is obviously continuous and defined on the closed set $[0, 10]$. It has, therefore, an absolute maximum (and minimum). The end-points $x = 0$, $x = 10$ give the minimum, whereas the maximum must then be found at an interior point where $A'(x) = 0$:

$$A'(x) = \frac{d}{dx}\left(x\sqrt{100-x^2}\right) = \sqrt{100-x^2} + x\,\frac{-2x}{2\sqrt{100-x^2}} = \sqrt{100-x^2}\left(1 - \frac{x^2}{100-x^2}\right).$$

This gives: $A'(x) = 0,\ (0 < x < 10) \;\Rightarrow\; 1 - \dfrac{x^2}{100-x^2} = 0 \;\Rightarrow\; x = 5\sqrt{2}.$

We also note that $A'(x) > 0$ for $0 < x < 5\sqrt{2}$ and $A'(x) < 0$ for $5\sqrt{2} < x < 10$.

This means that the maximum area is $A(5\sqrt{2}) = 50$.

9. We are given the function $f(x) = \dfrac{x^3}{x-1}$.

Find all critical points, local maxima/minima, inflection points, solutions of the equation $f(x) = 0$, and asymptotes of the function. Use this information to sketch the graph of f. *(5p)*

Answer:
Zeroes of the function: $f(x) = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$.

Furthermore, we note that f is differentiable for all x for which it is defined, that is $x \in D_f = (-\infty, 1) \cup (1, \infty)$. Taking the 1st derivative:

$$f'(x) = \frac{3x^2(x-1) - x^3}{(x-1)^2} = \frac{x^2(2x-3)}{(x-1)^2}.$$

In order to find any inflection points we also have to consider the 2nd derivative of f:

$$f'' = \frac{d}{dx}\Big(\frac{2x^3 - 3x^2}{(x-1)^2}\Big) = \frac{(6x^2-6x)(x-1)^2 - (2x^3-3x^2)\,2(x-1)}{(x-1)^4} =$$

$$\frac{2x(x-1)\Big(3(x-1)^2 - (2x^2-3x)\Big)}{(x-1)^4} = \frac{2x(x^2-3x+6)}{(x-1)^3}.$$

Note that $x^2 - 3x + 6 = (x - \frac{3}{2})^2 + \frac{15}{4} > 0$ for all $x \in \mathbb{R}$. This means that the sign of f'' is entirely determined by the other two factors.

The information above can be elucidated by the following chart (with obvious abbreviations):

		CP				CP	
x		0		1		3/2	
f'	$-$	0	$-$	UD	$-$	0	$+$
f''	$+$	0	$-$	UD	$+$	$+$	$+$
f	$\searrow$	IF	$\searrow$	UD	$\searrow$	min	$\nearrow$

In conclusion: f has a local minimum for $x = \dfrac{3}{2}$ and an inflection point at $(0, 0)$.

Using long division we may rewrite the function as $f(x) = x^2 + x + 1 + \dfrac{1}{x-1}$.

The expression above tells us that the f has the vertical asymptote $x = 1$, since $f(x) \to \pm\infty$ as $x \to 1$. The function has no horizontal or oblique asymptotes.

10. Calculate the volume of the body which is obtained by rotating the curve

$y = \dfrac{1}{\sqrt{1+4x^2}}, \quad 0 \le x \le \dfrac{1}{2}$ about the x-axis. *(5p)*

Answer:
The volume of the given solid of revolution is:

$$\int_0^{\frac{1}{2}} \pi\Big(\frac{1}{\sqrt{1+4x^2}}\Big)^2 dx = \pi \int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\, dx = [u = 2x]$$

$$= \frac{\pi}{2}\int_0^1 \frac{1}{1+u^2} = \frac{\pi}{2}\,\frac{1}{1+u^2}\Big|_0^1 = \frac{\pi}{2}\Big(\arctan 1 - \arctan 0\Big) = \frac{\pi}{2}\Big(\frac{\pi}{4} - 0\Big) = \frac{\pi^2}{8}.$$