Solutions Multivariable calculus, 2009-10-30.

1. Calculate the directional derivative of $f(x,y,z) = \ln\left(\dfrac{z}{x^2+y^2}\right)$

   at the point $P = (1,2,5)$ and in the direction $\boldsymbol{v} = (2,-1,2)$. *(2p)*

   Answer:
   Using the given formula and a normalized vector in the direction of $\boldsymbol{v} = (2,-1,2)$,
   $\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{3}(2,-1,2)$, we obtain:

   $$f'_{\boldsymbol{u}}(1,2,5) = \boldsymbol{u}\cdot\nabla f(1,2,5) = \boldsymbol{u}\cdot(f_x,f_y,f_z)_{(1,2,5)} = \boldsymbol{u}\cdot\left(-\frac{2x}{x^2+y^2}, -\frac{2y}{x^2+y^2}, \frac{1}{z}\right)_{(1,2,5)} =$$
   $$\tfrac{1}{3}(2,-1,2)\cdot(-\tfrac{2}{5},-\tfrac{4}{5},\tfrac{1}{5}) = \tfrac{2}{15}.$$

2. Find the following limit or show that it does not exist $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2-2xy)}{x^2+y^2}$. *(2p)*

   Answer:
   Compare the results of the following paths towards $(0,0)$:

   $(x,y) = (t,t)$ gives : $\dfrac{\sin(t^2+t^2-2t^2)}{t^2+t^2} = 0\ (t \neq 0)$,

   while $(x,y) = (t,0)$ gives : $\dfrac{\sin(t^2)}{t^2} \to 1$ as $t \to 0$. (see list of *Standard Limits*).

   These two different results constitute sufficient proof that the limit does *not* exist.

3. The equation $x^2z^3 + 2xz - \dfrac{y}{z} + 1 = 0$ implicitly defines a function $z = g(x,y)$

   for which $g(1,2) = -1$. Calculate $g_x(1,2)$ and $g_y(1,2)$. *(2p)*

   Answer:
   Taking the partial derivate with respect to x:

   $$\frac{\partial}{\partial x}\left(x^2z^3 + 2xz - \frac{y}{z} + 1\right) = 0 \Leftrightarrow 2xz^3 + 3x^2z^2z_x + 2z + 2xz_x + \frac{y}{z^2}\,z_x = 0 \Leftrightarrow$$
   $$z_x = -\frac{2z^3(xz^2+1)}{3x^2z^4+2xz^2+y} \Rightarrow g_x(1,2) = \frac{4}{7}.$$

   ... and with respect to y:

   $$\frac{\partial}{\partial y}\left(x^2z^3 + 2xz - \frac{y}{z} + 1\right) = 0 \Leftrightarrow 3x^2z^2z_y + 2xz_y - \frac{1}{z} + \frac{y}{z^2}\,z_y = 0 \Leftrightarrow$$
   $$z_y = \frac{z}{3x^2z^4+2xz^2+y} \Rightarrow g_y(1,2) = -\frac{1}{7}.$$

4. Find the center of mass $P = (\overline{x}, \overline{y})$ of a region D with constant density,

where $D = \{(x,y) \in \mathbb{R}^2 |\, x^2 \le y \le 1\}$. *(3p)*

Hint: $\overline{x} = \dfrac{1}{A_D} \iint\limits_D x\,dx\,dy$, $\overline{y} = \dfrac{1}{A_D} \iint\limits_D y\,dx\,dy$, where A_D is the area of the region D.

Answer:
Note that $x^2 \le y \le 1 \Rightarrow x^2 \le 1 \Leftrightarrow -1 \le x \le 1$.
Calculating the area of D:

$$A_D = \iint\limits_D dxdy = \int_{-1}^{1}\Big(\int_{x^2}^{1} dy\Big)\,dx = \int_{-1}^{1}(1-x^2)\,dx = \Big[x - \frac{1}{3}x^3\Big]_{-1}^{1} = \frac{4}{3}.$$

We then note that - due to symmetry - $\overline{x} = 0$, while $\overline{y}$ can be calculated as:

$$\overline{y} = \frac{1}{A_D}\iint\limits_D ydxdy = \frac{3}{4}\int_{-1}^{1}\Big(\int_{x^2}^{1} ydy\Big)dx = \frac{3}{4}\int_{-1}^{1}\Big[\frac{1}{2}y^2\Big]_{x^2}^{1}dx = \frac{3}{8}\int_{-1}^{1}(1-x^4)dx = \frac{3}{8}\Big[x - \frac{1}{5}x^5\Big]_{-1}^{1} = \frac{3}{5}.$$

This gives: $P = (\overline{x}, \overline{y}) = (0, \frac{3}{5})$.

5. Calculate the area of the region in the first quadrant bounded by the curves
$xy = 1$, $xy = 4$, $y = x$, and $y = 2x$. *(3p)*

Answer:
Denote the area as B. A change of variables will greatly simplify the calculation:

$u = xy$, $v = \dfrac{y}{x}$, giving: $1 \le u \le 4$, $1 \le v \le 2$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \Big(\dfrac{\partial(u,v)}{\partial(x,y)}\Big)^{-1} = \Big(\dfrac{2y}{x}\Big)^{-1} = \dfrac{1}{2v}$.

The area, A_B, can then be calculated as:

$$A_B = \iint\limits_B dx\,dy = \int_1^4\int_1^2 \frac{1}{2v}\,dv\,du = \frac{3}{2}\Big[\ln v\Big]_1^2 = \frac{3}{2}\ln 2.$$

Alternatively, the region B can be divided into 3 separate parts:

$$A_B = \int_{\frac{1}{\sqrt{2}}}^{1}\Big(\int_{\frac{1}{x}}^{2x} dy\Big)\,dx + \int_{1}^{\sqrt{2}}\Big(\int_{x}^{2x} dy\Big)\,dx + \int_{\sqrt{2}}^{2}\Big(\int_{x}^{\frac{4}{x}} dy\Big)\,dx = \frac{3}{2}\ln 2.$$

6. Find the absolute minimum and maximum values of $f(x,y,z) = x + 2y - 3z$
subject to the constraint $x^2 + 4y^2 + 9z^2 = 108$. *(3p)*

Answer:
Using the method of Lagrange multipliers and letting
$g(x,y,z) = x^2 + 4y^2 + 9z^2 - 108 = 0$ represent the constraint :

$$\begin{array}{lcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \\ f_z &=& \lambda g_z \end{array} \Leftrightarrow \begin{array}{rcl} 1 &=& 2\lambda x \\ 2 &=& 8\lambda y \\ -3 &=& 18\lambda z \end{array} \Rightarrow \begin{array}{rcl} 0 &=& 6\lambda x + 18\lambda z \\ 0 &=& 24\lambda y + 36\lambda z \end{array} \Leftrightarrow$$

$(\lambda = 0 \lor x = -3z) \land (\lambda = 0 \lor y = -\frac{3}{2}z)$.

The 'reduced' set of 2 equations has been obtained by linear combinations of the original 3 equations in the Gaussian manner. Obviously, $\lambda = 0$ does not satisfy the original equations. Hence, we get the following result from the Lagrangian equations: $(x,y,z) = (-3z, -\frac{3}{2}z, z)$. This we insert into the constraint:

$$(-3z)^2 + 4\Big(-\frac{3}{2}z\Big)^2 + 9z^2 = 108 \Leftrightarrow 27z^2 = 108 \Leftrightarrow z = -2 \lor z = 2.$$

Comparison and conclusion:

$f(-6,-3,2) = -18$, $f(6,3,-2) = 18$, $f_{min} = -18$, $f_{max} = 18$.

7. Find the absolute minimum and maximum values of $g(x, y) = 9xy^2e^{-2x-3y}$ on the set $\Delta = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le 2\}$. *(5p)*

Answer:
First finding interior critical points:

$$\begin{array}{rcl} g_x & = & 0 \\ g_y & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 9y^2e^{-2x-3y}(1-2x) & = & 0 \\ 9xye^{-2x-3y}(2-3y) & = & 0 \end{array} \Leftrightarrow (y = 0 \ \vee\ x = \tfrac{1}{2}) \wedge (x = 0 \ \vee\ y = 0 \ \vee y = \tfrac{2}{3}).$$

The only *interior* point is then $(\frac{1}{2}, \frac{2}{3})$.

Next we consider the values of g along the four sides of the square that constitutes the boundary of Δ :

1: $0 \le x \le 2,\ y = 0 :\ \ g(x, 0) = 0$. Interesting points: $(0, 0),\ (2, 0)$.

2: $0 \le y \le 2,\ x = 0 :\ \ g(0, y) = 0$. Additional interesting points: $(0, 2)$.

3: $0 \le x \le 2,\ y = 2 :\ g(x, 2) = 36e^{-6}\, xe^{-2x}$.

$$\frac{d}{dx}\Big(36e^{-6}\, xe^{-2x}\Big) = 0 \ \Rightarrow\ x = \frac{1}{2}.$$

Additional interesting points : $(\frac{1}{2}, 2),\ (2, 2)$.

4: $0 \le y \le 2,\ x = 2 :\ \ g(2, y) = 18e^{-4}\, y^2e^{-3y}$.

$$\frac{d}{dy}\Big(18e^{-6}\, y^2e^{-3y}\Big) = 0 \ \Rightarrow\ y = 0 \ \vee\ y = \frac{2}{3}.$$

Additional interesting point : $(2, \frac{2}{3})$.

Comparison of the function values at the points found above:

$g(\frac{1}{2}, \frac{2}{3}) = 2e^{-3},\ \ g(0, 0) = g(2, 0) = g(2, 0) = 0,\ \ g(\frac{1}{2}, 2) = 18e^{-7},\ \ g(2, 2) = 72e^{-10},\ \ g(2, \frac{2}{3}) = 8e^{-6}$.

In ascending order: $0 < 72e^{-10} < 18e^{-7} < 8e^{-6} < 2e^{-3}$.

If this is not obvious, note that the last inequality is true since $8e^{-6} < 2e^{-3} \Leftrightarrow 4 < e^3$.

The other two inequalities may be demonstrated in a similar manner.

This means that: $f_{min} = 0,\ f_{max} = 2e^{-3}$.

8. Calculate $\displaystyle\iint_S x^2z^2\, dS$, where $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2},\ 1 \le x^2 + y^2 \le 2\}$. *(5p)*

Answer:
We first note that: $\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \Big(\dfrac{2x}{2\sqrt{x^2+y^2}}\Big)^2 + \Big(\dfrac{2y}{2\sqrt{x^2+y^2}}\Big)^2} = \sqrt{2}$.

Furthermore, the projection of the surface S on to the xy - plane is given by

$S_{xy}:\ 1 \le x^2 + y^2 \le 2 \Leftrightarrow 1 < r < \sqrt{2}$ (in polar coordinates).

Also, since the projection is a region between two circular disks : $0 \le \theta \le 2\pi$.

The surface integral can then be performed as:

$$\iint_S x^2z^2\, dS = \iint_{S_{xy}} x^2 \left(\sqrt{x^2+y^2}\right)^2 \sqrt{2}\, dx\, dy =$$

$$\sqrt{2} \int_0^{2\pi}\int_1^{\sqrt{2}} r^2\, cos^2\theta\, r^2\, rdr\, d\theta = \sqrt{2} \int_0^{2\pi} \cos^2\theta\, d\theta \int_1^{\sqrt{2}} r^5\, dr =$$

$$\sqrt{2}\left[\frac{1}{6}\, r^6\right]_1^{\sqrt{2}} \int_0^{2\pi} \frac{1}{2}\Big(1 + \cos 2\theta\Big)\, d\theta = \frac{7\sqrt{2}\,\pi}{6}.$$

9. Calculate $\iiint\limits_K \frac{z}{1+\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$, $K = \{(x,y,z)\,\epsilon\,\mathbb{R}^3\,|\,x^2+y^2+z^2 \le 1,\ z \ge 0\}$. *(5p)*

Answer:
Using spherical coordinates and noting that K is a hemisphere with radius $R = 1$ lying above the xy - plane, we obtain the limits: $0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 1.$

The integral can then be calculated as:

$$\iiint\limits_K \frac{z}{1+\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} \frac{\rho\,\cos\phi}{\rho+1}\,\rho^2 d\rho\ \sin\phi\,d\phi\,d\theta =$$

$$2\pi\int_0^{\frac{\pi}{2}} \cos\phi\ \sin\phi\,d\phi \int_0^{1}\frac{\rho^3}{\rho+1}\,d\rho = 2\pi\int_0^{\frac{\pi}{2}}\frac{1}{2}\ \sin 2\phi\,d\phi\int_0^{1}\Big(\rho^2-\rho+1-\frac{1}{\rho+1}\Big)\,d\rho =$$

$$\pi\left[-\frac{1}{2}\cos 2\phi\right]_0^{\frac{\pi}{2}}\left[\frac{1}{3}\rho^3-\frac{1}{2}\rho^2+\rho-\ln(\rho+1)\right]_0^1 = \pi\Big(\frac{5}{6}-\ln 2\Big).$$