Laboration PM
Numerical methods
Master programme of Financial Mathematics
Halmstad University

Simulation

Part A: Random number generators

1. Choose values a, b and x_0 such that $\{x_n\}$ is a random number generator with period 100 defined by
$$x_{n+1} \;=\; a + bx_n \pmod{101}$$
for $n = 0, 1, 2, \ldots, 1000$.

2. Make another sequence $\{y_n\}$ of pseudo-random numbers generated by values c, d and y_0 and the recursion algorithm
$$y_{n+1} \;=\; c + dy_n \pmod{103}$$
for $n = 0, 1, 2, \ldots, 1000$. The period of the the sequences $\{x_n\}$ and $\{y_n\}$ is defined by $p_x = \min\{n \geq 1 : x_n = x_0\}$ and $p_y = \min\{n \geq 1 : y_n = y_0\}$, but how should the period of the sequence $\{x_n + y_n\}$ be defined.

3. Choose c, d and y_0 so that the period of $\{x_n + y_n\}$ is at least 1000. Plot a histogram of $\{x_n + y_n\}$. What is your conclusion?

4. Assume U_1 and U_2 are random variables such that $U_1 \perp U_2$, $U_1 \in U[0, 1]$ and $U_2 \in U[0, 1]$. Prove that the distribution function of $U_1 + U_2$ is
$$F_{U_1+U_2}(u) \;=\; \begin{cases} 0 & \text{if } u < 0 \\ u^2/2 & \text{if } 0 \leq u < 1 \\ 2u - u^2/2 - 1 & \text{if } 1 \leq u < 2 \\ 1 & \text{if } u \geq 2 \end{cases}$$

5. Suppose X is a continuous random variable with strictly continuous distribution function F. Prove that $F(X) \in U[0, 1]$.

6. Make an appropriate transformation of the sequence $\{x_n + y_n\}$ into a sequence $\{z_n\}$ which look more like uniform deviates.

7. Test on level 5% of significance whether the sequence of pseudo-uniform deviates follows a uniform distribution (or rather if one can *not* prove with significance level 5% that the sequence is *not* uniformly distributed) by performing a frequency test.

Part B: Simulating deviates from multivariate normal distribution

1. Choose the `R` routine `runif` to simulate 100 observations of a random variable uniformly distributed on $[0, 1]$. Plot the result using the routine `plot`.

2. Use the Box-Müller method to transform these into observations of 100 independent standard normally distributed variables. Plot the result using the routine `plot` and illustrate the fit to the normal distribution by the routine `qqplot`.

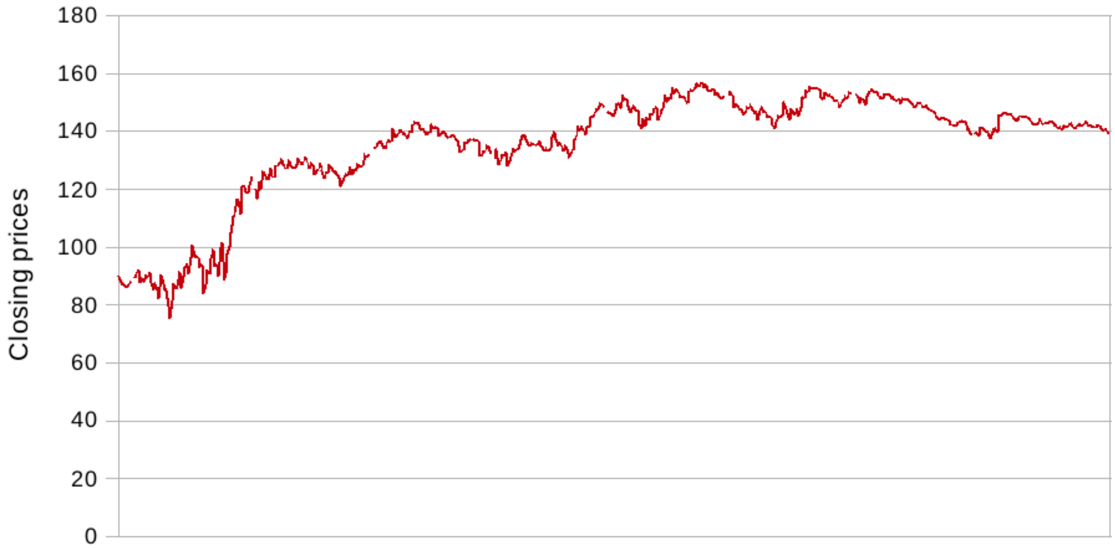
3. Now assume $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{100})$ is a multivariate normally distributed random variable with $E(\boldsymbol{Z}) = \mathbf{0}$ and a covariance matrix C with entries

$$c_{ij} = Cov(X_i, X_j) = \frac{1}{(i-j)^2 + 1} \quad i, j \in \{1, 2, \ldots, 100\}$$

   Use the simulation in 2. to simulate one observation of the random variable $\boldsymbol{Z}$.

Part C: Estimation of a probability of an ARARCH model

Consider the time series data of closing prices of the OMX SPI-index of the Stockholm stock exchange market during March 1 2006 – March 1 2009.



These data are available at Yahoo finance:
`http://www.yahoo.com` →`Finance` →`Market summary: Europe` →`View more indices` →`OMXSPI` →`Historical prices` [Select] `From: March 1 2006, To: March 1 2009, Get prices` [Scroll down, Select] `Download to spreadsheet`.

Direct URL: `http://ichart.finance.yahoo.com/table.csv?s=%5EOMXSPI&a=02&b=1&c=2006&d=02&e=1&f=2009&g=d&ignore=.csv`

Let us denote this time series data by $\{x_n : n = 1, 2, \dots, 535\}$.

The process $\{X_n : n \in \mathbb{Z}\}$ could be modelled as $X_n = \mu + Y_n$ where $\mu \in \mathbb{R}$ is a *baseline* and $\{Y_n : n \in \mathbb{Z}\}$ is an $ARARCH(1, 1)$ process, i.e. it is defined recursively by

$$X_t = aX_{t-1} + \sigma_t \epsilon_t$$

where $\sigma_n^2 = b + cX_{n-1}^2$, $\{\epsilon_n\}$ is (standard normally distributed) white noise, and $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $c \in \mathbb{R}^+$ are parameters. The goal is now to estimate the probability that $P(X_{536} \leq 138 \,|\, \mathcal{F}_{535})$.

1. Estimate $E(X_n)$, $E(X_n^2)$, $E(X_n^4)$ and $E(X_n X_{n-1})$ from the data.

2. Simulate $\{X_n\}$ with some values of μ, a, b and c and calculate the estimates of $E(X_n)$, $E(X_n^2)$, $E(X_n^4)$ and $E(X_n X_{n-1})$ from that simulation.

3. Change the parameter values μ, a, b and c such that the estimates of $E(X_n)$, $E(X_n^2)$, $E(X_n^4)$ and $E(X_n X_{n-1})$ are approximately equal to the estimates based on the data set.

4. Finally use the estimated parameters to estimate $P(X_{536} \leq 138 \,|\, \mathcal{F}_{535})$ by rejection sampling.