Signal Processing

Collection of Formulas

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Contents

1 Basic Relationships and Concepts

1.1 Standard Signals

1.1.1 Continuous Functions

T is the period time for a perodic function $f(t)$, where t [s] is the time variable
Angular frequency $\Omega = \frac{2\pi}{T} = 2\pi F$ [rad/s], F [Hz] is the frequency.

a) Impulse Function $\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

$\int_{-\infty}^{\infty} \delta(t)dt = 1$

$x(t)\delta(t-a) = x(t-a)\delta(t)$

$\delta(\frac{1}{a}) = a\delta(t)$

$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$

b) Step Function $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

c) Ramp Function $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$

d) Rectangular Pulse $p(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$

e) Sinc Function $\text{sinc } t = \frac{\sin \pi t}{\pi t}$

f) Periodic Function $f(t) = f(t+T)$ where $T \neq 0$

g) Periodic Sinc Function $\text{diric}(t, T) = \frac{\sin\left(\frac{Tt}{2}\right)}{T \sin\left(\frac{t}{2}\right)}$

h) Complex Periodic Signal $e^{st} = e^{\sigma t} e^{j\Omega t}$

i) Complex undamped per. signal $e^{j\Omega t} = \cos \Omega t + j \sin \Omega t$

1.1.2 Discrete Time Functions

N is the period time (integer) for a periodic function $f(n)$, where n is a discrete time index (integer). Normalized angular frequency $\omega = \frac{2\pi}{N} = 2\pi f$ [rad], f is normalized frequency.

a) Unit Pulse $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

b) Unit Step $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

c) Ramp $r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

d) Rectangular Pulse $p(n) = \begin{cases} 1 & |n| < n_1 \\ 0 & |n| > -n_1 \end{cases}$

e) Sinc Function $\text{sind}_N(n) = \dfrac{\sin\left(\frac{Nn}{2}\right)}{N\sin\left(\frac{n}{2}\right)}$

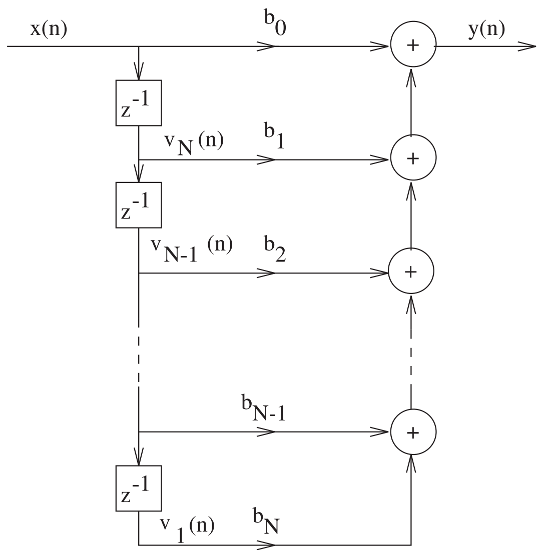
f) Periodic Signal $f(n) = f(n+N)$ where $N \neq 0$

g) Complex periodic signal $e^{sn} = e^{\sigma n}e^{j\omega n}$

h) Complex undamped per. signal $e^{j\omega n} = \cos\omega n + j\sin\omega n$

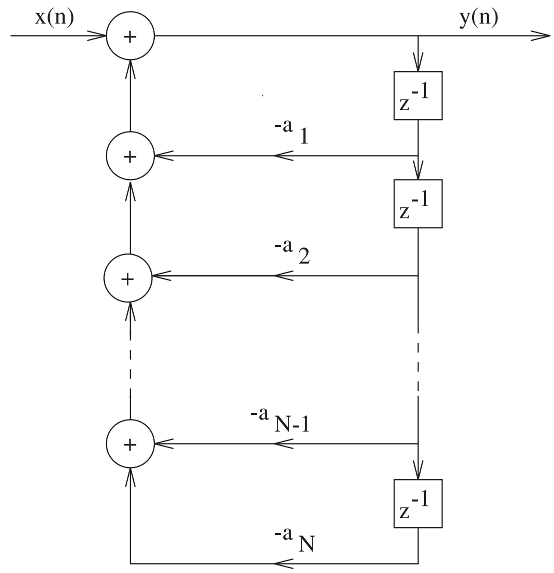
1.2 SISO systems (Single input, single output) Discrete Time Systems

1.2.1 FIR Filters



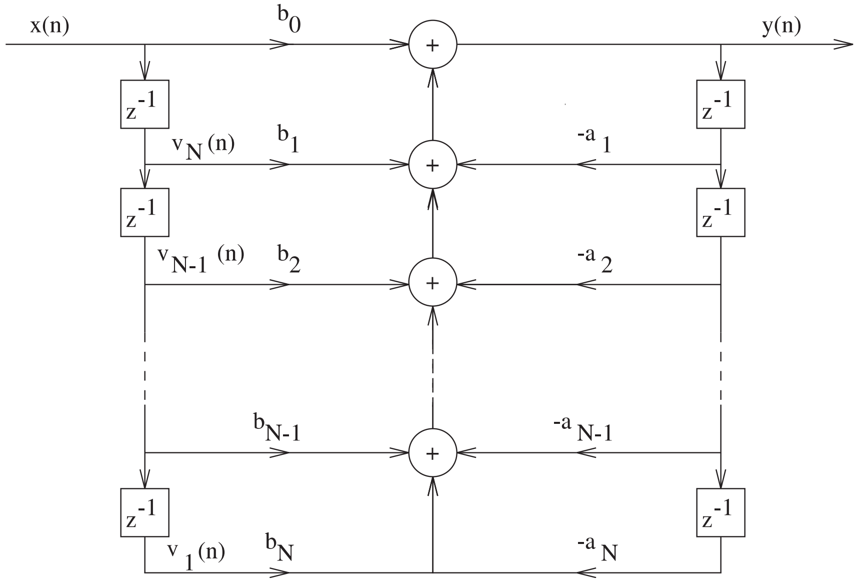
$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

1.2.2 IIR Filters



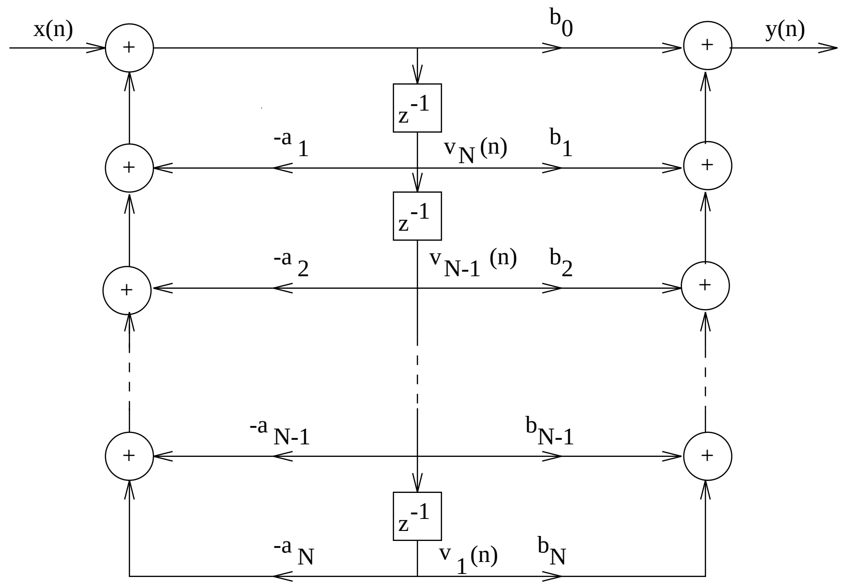
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k)$$

1.2.3 IIR Canonical form - Direct Form I



$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

1.2.4 IIR Canonical form - Direkt Form II



$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

1.3 Some Methods of Calculation

1.3.1 Convolution

$$y(n) = h * x = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

1.3.2 State-Space Model of Difference Equation

If

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

the State-Space Model is given by

$$\begin{cases} \mathbf{v}(n+1) &= \mathbf{F}\mathbf{v}(n) + \mathbf{q}\cdot x(n) \\ y(n) &= \mathbf{g^T}\mathbf{v}(n) + d\cdot x(n) \end{cases}$$

where

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ -a_k & -a_{k-1} & \dots & -a_2 & -a_1 \end{pmatrix} \quad ; \quad \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\mathbf{g^T} = (b_k, \dots, b_2, b_1) - b_0(a_k, \dots, a_2, a_1) \quad ; \quad d = b_0$$

1.3.3 State-Space Equation

a) Direct Solution

$$y(n) = \mathbf{g^T} \cdot \mathbf{F}^n \mathbf{v}(0) + \sum_{k=0}^{n-1} \mathbf{g^T} \cdot \mathbf{F}^{n-1-k} \mathbf{q} x(k) u(n-1) + dx(n)$$

b) Impulse Function

$$h(n) = \mathbf{g^T} \cdot \mathbf{F}^{n-1} \mathbf{q} u(n-1) + d\delta(n)$$

c) System Function

$$\mathcal{H}(z) = \mathbf{g^T}[z\mathbf{I} - \mathbf{F}]^{-1}\mathbf{q} + d$$

1.3.4 System Function

$$\mathcal{H}(z) = \frac{\mathcal{Y}(z)}{\mathcal{X}(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

1.4 Analogue Sinusoidal Signal through Linear, Causal Filter

1.4.1 Complex, Non-causal Input Signal

$$x(t) = e^{j\Omega_0 t} = (\cos(\Omega_0 t) + j\ \sin(\Omega_0 t)) \quad -\infty < t < \infty$$

$$y(t) = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)e^{j\Omega_0(t-\tau)}d\tau = \underbrace{H(s)|_{s=j\Omega_0}\ e^{j\Omega_0 t}}_{\text{stationary}}$$

1.4.2 Complex, Causal Input Signal

$$x(t) = e^{j\Omega_0 t}u(t) = (\cos(\Omega_0 t) + j\ \sin(\Omega_0 t))\ u(t); \qquad X(s) = \frac{1}{s - j\Omega_0}$$

$$Y(s) = H(s)X(s) = \frac{T(s)}{N(s)}\ \frac{1}{s-j\Omega_0} = \underbrace{\frac{T_1(s)}{N(s)}}_{\text{transient}} + \underbrace{H(s)|_{s=j\Omega_0}\ \frac{1}{s-j\Omega_0}}_{\text{stationary}}$$

$$y(t) = \text{transient} + \underbrace{H(s)|_{s=j\Omega_0}\ e^{j\Omega_0 t}}_{\text{stationary}}$$

1.4.3 Real, Non-causal Input Signal

$$x(t) = Re\{e^{j\Omega_0 t}\} = \cos(\Omega_0 t) \quad -\infty < t < \infty$$

$$y(t) = \int_{\tau=0}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{\tau=0}^{\infty} h(\tau)\frac{1}{2}(e^{j\Omega_0(t-\tau)} + e^{-j\Omega_0(t-\tau)})d\tau =$$

$$= \underbrace{|H(s)|_{s=j\Omega_0}\ \cos(\Omega_0 t + arg\{H(s)|_{s=j\Omega_0}\})}_{\text{stationary}}$$

1.4.4 Real, Causal Input Signal

$$x(t) = Re\{e^{j\Omega_0 t}\}\ u(t) = \cos(\Omega_0 t)\ u(t); \qquad X(s) = \frac{s}{s^2 + \Omega_0^2}$$

$$Y(s) = H(s)X(s) = \frac{T(s)}{N(s)}\ \frac{s}{s^2+\Omega_0^2} = \underbrace{\frac{T_1(s)}{N(s)}}_{\text{transient}} + \underbrace{\frac{C_1 s + C_0}{s^2+\Omega_0^2}}_{\text{stationary}}$$

$$H(s)|_{s=j\Omega_0} = A\ e^{j\theta};\ C_1 = A\cos(\theta);\ C_0 = -A\Omega_0\sin\theta$$

$$y(t) = \text{transient} + \underbrace{C_1 \cos(\Omega_0 t) + \frac{C_0}{\Omega_0} \sin(\Omega_0 t)}_{\text{stationary}} =$$

$$= \text{transient} + \underbrace{|H(s)|_{s=j\Omega_0} \; \cos(\Omega_0 t + arg\{H(s)|_{s=j\Omega_0}\})}_{\text{stationary}}$$

1.5 Discrete Time Sinusoidal Signal through Linear, Causal Filter

1.5.1 Complex, Non-Causal Input Signal

$$x(n) = e^{j\omega_0 n} = (\cos(\omega_0 n) + j\ \sin(\omega_0 n)) \quad -\infty < n < \infty$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} h(k)e^{j\omega_0(n-k)} = \underbrace{H(z)|_{z=e^{j\omega_0}}\ e^{j\omega_0 n}}_{\text{stationary}}$$

1.5.2 Complex, Causal Input Signal

$$x(n) = e^{j\omega_0 n}u(n) = (\cos(\omega_0 n) + j\ \sin(\omega_0 n))\ u(n); \qquad X(z) = \frac{1}{1 - e^{j\omega_0}z^{-1}}$$

$$Y(z) = H(z)X(z) = \frac{T(z)}{N(z)}\ \frac{1}{1 - e^{j\omega_0}z^{-1}} = \underbrace{\frac{T_1(z)}{N(z)}}_{\text{transient}} + \underbrace{H(z)|_{z=e^{j\omega_0}}\ \frac{1}{1 - e^{j\omega_0}z^{-1}}}_{\text{stationary}}$$

$$y(n) = \text{transient} + \underbrace{H(z)|_{z=e^{j\omega_0}}\ e^{j\omega_0 n}}_{\text{stationary}}$$

1.5.3 Real, Non-Causal Input Signal

$$x(n) = Re\{e^{j\omega_0 n}\} = \cos(\omega_0 n) \quad -\infty < n < \infty$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} h(k)\frac{1}{2}(e^{j\omega_0(n-k)} + e^{-j\omega_0(n-k)}) =$$

$$= \underbrace{|H(z)|_{z=e^{j\omega_0}}\ \cos(\omega_0 n + arg\{H(z)|_{z=e^{j\omega_0}}\})}_{\text{stationary}}$$

1.5.4 Real, Causal Input Signal

$$x(n) = Re\{e^{j\omega_0 n}\}\ u(n) = \cos(\omega_0 n)\ u(n); \qquad X(z) = \frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}$$

$$Y(z) = H(z)X(z) = \frac{T(z)}{N(z)}\ \frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}} = \underbrace{\frac{T_1(z)}{N(z)}}_{\text{transient}} + \underbrace{\frac{C_0 + C_1 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}}_{\text{stationary}}$$

$$H(z)|_{z=e^{j\omega_0}} = A\ e^{j\theta};\ C_0 = A\cos(\theta);\ C_1 = -A(\sin\omega_0 \sin\theta + \cos\omega_0 \cos\theta)$$

$$y(n) = \text{transient} + \underbrace{C_0 \cos(\omega_0 n) + \frac{C_1 + C_0 \cos(\omega_0)}{\sin(\omega_0)} \sin(\omega_0 n)}_{\text{stationary}} =$$

$$= \text{transient} + \underbrace{|H(z)|_{z=e^{j\omega_0}}\ \ \cos(\omega_0 n + arg\{H(z)|_{z=e^{j\omega_0}}\})}_{\text{stationary}}$$

1.6 Fourier Series Expansion

For table see Appendix

1.6.1 Continuous Time

A periodic function with period T_0 , i.e. $f(t) = f(t - T_0)$, can be expressed in the form of a series expansion according to

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \; e^{j2\pi k F_0 t}$$

where

$$c_k = \frac{1}{T_0}\int_{T_0} f(t) e^{-j2\pi k F_0 t} dt \; ; F_0 = \frac{1}{T_0}$$

If $f(t)$ is real this can also be expressed as

$$\begin{aligned} f(t) &= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) = \\ &= a_0 + \sum_{k=1}^{\infty} a_k \cos 2\pi k F_0 t - b_k \sin 2\pi k F_0 t \end{aligned}$$

where

$$\begin{aligned} a_0 &= c_0 = \frac{1}{T_0}\int_{T_0} f(t) dt \\ a_k &= 2|c_k| \cos\theta_k = \frac{2}{T_0}\int_{T_0} f(t)\cos(2\pi k F_0 t) dt \\ b_k &= 2|c_k| \sin\theta_k = \frac{-2}{T_0}\int_{T_0} f(t)\sin(2\pi k F_0 t) dt \end{aligned}$$

The power is given by (Parseval's Relation)

$$P = \frac{1}{T_0}\int_{T_0} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

In addition, for real signals

$$P = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

1.6.2 Discrete Time

A periodic function with the period N, i.e. $f(n) = f(n - N)$, can be expressed as a series expansion according to

$$f(n) = \sum_{k=0}^{N-1} c_k \; e^{j2\pi k \; n/N}$$

where

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} f(n)\ e^{-j2\pi k\ n/N}, \quad k = 0, \ldots, N-1$$

The series expansion is often written using DTFS (Discrete-Time Fourier Series). If $f(n)$ is real this can also be expressed as

$$\begin{aligned} f(n) &= c_0 + 2\sum_{k=1}^{L} |c_k| \cos\left(2\pi\ \frac{kn}{N} + \theta_k\right) = \\ &= a_0 + \sum_{k=1}^{L} \left(a_k \cos\left(2\pi\ \frac{kn}{N}\right) - b_k \sin\left(2\pi\ \frac{kn}{N}\right)\right) \end{aligned}$$

where

$$\begin{aligned} a_0 &= c_0 \\ a_k &= 2|c_k| \cos(\theta_k) \\ b_k &= 2|c_k| \sin(\theta_k) \\ L &= \begin{cases} \frac{N}{2} & \text{if } N \text{ even} \\ \\ \frac{N-1}{2} & \text{if } N \text{ odd} \end{cases} \end{aligned}$$

The power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |f(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

and the energy over one period is given by

$$E_N = \sum_{n=0}^{N-1} |f(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

1.7 Fourier Transformer

Continuous time signal:

$$\begin{cases} X_a(F) &= \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt \\ \\ x_a(t) &= \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} dF \end{cases}$$

Discrete time signal:

$$\begin{cases} X(f) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \\ \\ x(n) &= \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df \end{cases}$$

1.8 Discrete Fourier Transform (DFT)

1.8.1 Definition

$$X_k = DFT\ (x_n) = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1 \quad \text{Transform}$$
$$x_n = IDFT\ (X_k) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \quad n = 0, 1, \ldots, N-1 \quad \text{Inversion}$$

Note:

$$\sum_{n=0}^{N-1} e^{j2\pi\ \frac{k-k_0}{N}\cdot n} = N \cdot \delta(k - k_0, (\text{modulo } N))$$

1.8.2 Circular Convolution

$$x_n\ N\ y_n = \sum_{\ell=0}^{N-1} x_\ell y_{n-\ell} \stackrel{\text{DFT}}{\longleftrightarrow} X_k Y_k \quad \text{Circular convolution}$$

where $\sum$ denotes circular convolution. This means that the sequences x_n and y_n should be repeated periodically before the summation. I.e. outside the interval $n = 0, 1, \ldots, N-1$, $x_{n-\ell N} = x_n$ and $y_{n-\ell N} = y_n$ ($\ell =$ integer) In other words, the index is calculated modulo N. Circular convolution is also denoted $x(n)\ *\ y(n)$.

1.8.3 Non-Circular Convolution using DFT

If $x(n) = 0$ for $n \neq [0, L-1]$ and $y(n) = 0$ for $n \neq [0, M-1]$ then $x * y = 0$ for $n \neq [0, N-1]$ where $N \geq L + M - 1$.
The convolution can be calculated as

$$x * y = \begin{cases} x \circledast y = \text{IDFT}(X_k Y_k) & n = 0, 1, \ldots, N-1 \\ 0 & \text{Else} \end{cases}$$

where

$$\begin{aligned} X_k &= \text{DFT}(x(n)) \\ Y_k &= \text{DFT}(y(n)) \end{aligned}$$

1.8.4 Relation to the Fourier Transform $X(f)$:

$$\begin{aligned} X(k/N) &= X_k = DFT(x(n)) \text{ if } x(n) = 0 \text{ for } n \neq [0, N-1] \\ X(k/N) &= X_k = DFT(x_p(n)) \text{ generally } x(n) \text{ where } x_p(n) = \sum_{\ell=-\infty}^{\infty} x(n - \ell N) \end{aligned}$$

1.8.5 Relation to Fourier Series

$$X\left(\frac{k}{N}\right) = X_k = DFT(x(n)) = N \cdot c_k$$

if

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1$$

where

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi\frac{nk}{N}} \quad -\infty < n < \infty$$

and

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi\frac{nk}{N}} \quad k = 0, 1, \ldots, N-1$$

1.8.6 Parseval's Theorem

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y^*(k)$$

1.9 Some Window Functions and their Fourier Transforms

i) The window functions are centered around the origin (odd filter length M), i.e. the functions are not equal to zero only for $-(M-1)/2 \leq n \leq (M-1)/2$

Rectangular window:

$$w_{\text{rect}}(n) = 1$$

$$W_{\text{rect}}(f) = M \cdot \frac{\sin(\pi f M)}{M \sin(\pi f)}$$

Bartlett window (Triangular window):

$$w(n) = 1 - \frac{|n|}{(M-1)/2}$$

$$W(f) = \frac{M}{2} \left(\frac{\sin \frac{\pi f M}{2}}{\frac{M}{2} \sin(\pi f)} \right)^2 \approx \frac{2}{M} W_{\text{rect}}^2 \left(\frac{f}{2} \right) \text{ if } f \text{ is small}$$

Hanning Window:

$$w(n) = 0.5 + 0.5 \cos \left(\frac{2\pi n}{M-1} \right)$$

$$\begin{aligned} W(f) = {} & 0.5\, W_{\text{rect}}(f) + \\ & +0.25\, W_{\text{rect}} \left(f - \frac{1}{M-1} \right) + \\ & +0.25\, W_{\text{rect}} \left(f + \frac{1}{M-1} \right) \end{aligned}$$

Hamming Window:

$$w(n) = 0.54 + 0.46 \cos \left(\frac{2\pi n}{M-1} \right)$$

$$\begin{aligned} W(f) = {} & 0.54\, W_{\text{rect}}(f) + \\ & +0.23\, W_{\text{rect}} \left(f - \frac{1}{M-1} \right) + \\ & +0.23\, W_{\text{rect}} \left(f + \frac{1}{M-1} \right) \end{aligned}$$

Blackman window:

$$w(n) = 0.42 + 0.5 \cos \frac{2\pi n}{M-1} + 0.08 \cos \frac{4\pi n}{M-1}$$

$$\begin{aligned} W(f) = {} & 0.42\, W_{\text{rect}}(f) + \\ & +0.25\, W_{\text{rect}} \left(f - \frac{1}{M-1} \right) + \end{aligned}$$

$$+0.25\ W_{\text{rect}}\left(f+\frac{1}{M-1}\right)+$$
$$+0.04\ W_{\text{rect}}\left(f-\frac{2}{M-1}\right)+$$
$$+0.04\ W_{\text{rect}}\left(f+\frac{2}{M-1}\right)$$

Kaiser window:

$$w(n)=\frac{I_0\left(\beta\sqrt{1-[2n/(M-1)]^2}\right)}{I_0(\beta)}$$

$$I_0(x)=1+\sum_{k=1}^{L}\left[\frac{(x/2)^k}{k!}\right]^2$$

2 Sampling Analogue Signals

2.1 Sampling and Reconstruction

Sampling Theorem

If $x_a(t)$ is bandlimited, i.e. $X_a(F) = 0$ for $|F| \geq 1/2T$ then

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin \frac{\pi}{T} (t - nT)}{\frac{\pi}{T} (t - nT)}$$

Sampling Frequency $F_s = 1/T$.

Sampling

$$\begin{aligned} x(n) &= x_a(nT) \; ; \quad T = \frac{1}{F_s} \\ X(f) &= X\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \\ \Gamma(f) &= \Gamma\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} \Gamma_a(F - kF_s) \end{aligned}$$

Reconstruction (ideal)

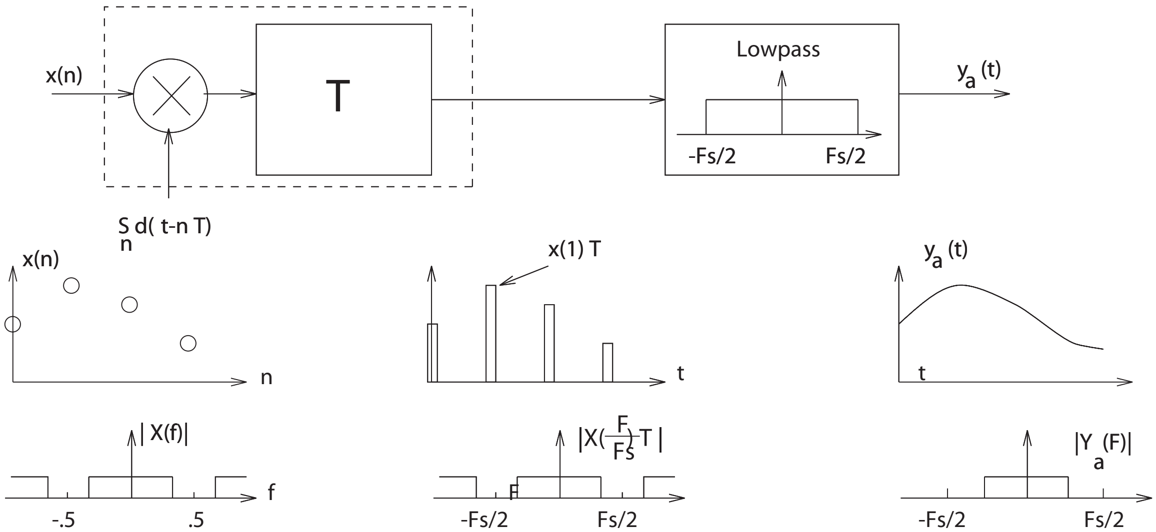
$$\begin{aligned} x_a(t) &= \sum_{n=-\infty}^{\infty} x(n) \frac{sin \frac{\pi}{T} (t - nT)}{\frac{\pi}{T} (t - nT)} \\ X_a(F) &= \frac{1}{F_s} X\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2} \\ \Gamma_a(F) &= \frac{1}{F_s} \Gamma\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2} \end{aligned}$$

Reconstruction using Sample-and-Hold

$$\begin{aligned} X_a(F) &= \frac{1}{F_s} X\left(\frac{F}{F_s}\right) \cdot \frac{sin(\pi FT)}{\pi FT} \, e^{-j2\pi F \frac{T}{2}} \cdot H_{LP}(F) \\ \Gamma_a(F) &= \frac{1}{F_s} \Gamma\left(\frac{F}{F_s}\right) \left|\frac{\sin(\pi FT)}{\pi FT}\right|^2 \cdot |H_{LP}(F)|^2 \end{aligned}$$

Block Scheme describing D/A conversion

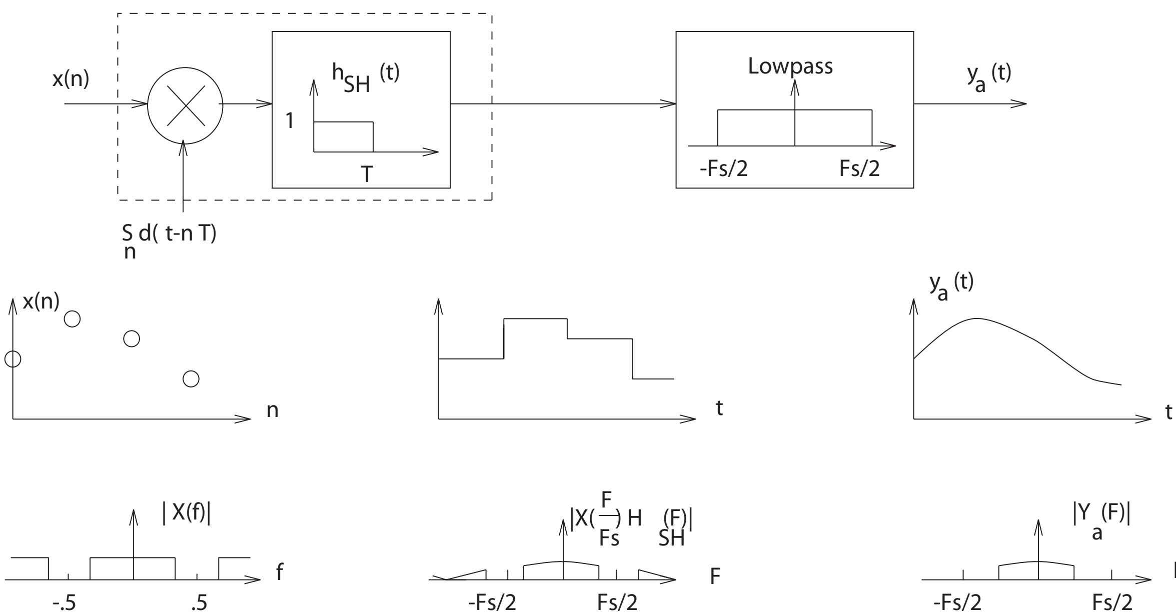
Ideal reconstruction



$$y_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{sin \ \frac{\pi}{T} \ (t-nT)}{\frac{\pi}{T} \ (t-nT)}$$

$$Y_a(F) = \frac{1}{F_s} \ X\left(\frac{F}{F_s}\right) \quad |F| \leq \frac{F_s}{2}$$

Reconstruction using Sample-and-Hold



$$Y_a(F) = \frac{1}{F_s} \ X\left(\frac{F}{F_s}\right) \cdot \frac{sin(\pi FT)}{\pi FT} \ e^{-j2\pi F \ \frac{T}{2}} \cdot H_{LP}(F)$$

2.2 Measures of Distortion

2.2.1 Folding distortion when sampling

Spectrum after anti-aliasing filter:

$$\Gamma_{in}(F)$$

Folding distortion:

$$D_A = 2 \cdot \int_{F_s - F_p}^{\infty} \Gamma_{in}(F) dF$$

Effective signal power:

$$D_s = 2 \int_0^{F_p} \Gamma_{in}(F) dF$$

where $0 \leq F_p \leq F_s/2$

Signal distortion relationship:

$$\text{A:} \quad SDR_A = \frac{D_S}{D_A} = \frac{\int_0^{F_p} \Gamma_{in}(F) dF}{\int_{F_s - F_p}^{\infty} \Gamma_{in}(F) dF}$$

$$\text{B:} \quad SDR_A^0 = \min_{|F| \leq F_p} \; \frac{\Gamma_{in}(F)}{\Gamma_{in}(F_s - F)}$$

If the spectrum is monotonously decreasing

$$SDR_A^0 = \frac{\Gamma_{in}(F_p)}{\Gamma_{in}(F_s - F_p)}$$

2.2.2 Periodic Distortion when Reconstructing

Periodic Distortion:

$$D_P = 2 \cdot \int_{F_s/2}^{\infty} \Gamma_{ut}(F) dF$$

Effective signal power:

$$D_S = 2 \cdot \int_0^{F_s/2} \Gamma_{ut}(F) dF$$

Signal Distortion Relationship:

$$\text{A:} \quad SDR_P = \frac{D_S}{D_P} = \frac{\int_0^{F_s/2} \Gamma_{ut}(F) dF}{\int_{F_s/2}^{\infty} \Gamma_{ut}(F) dF}$$

$$\text{B:} \quad SDR_P^0 = \min_{|F| < F_s/2} \; \frac{\Gamma_{ut}(F)}{\Gamma_{ut}(F_s - F)}$$

A good estimate is often given by

$$SDR_P^0 = \frac{\Gamma_{ut}(F_p)}{\Gamma_{ut}(F_s - F_p)}$$

where F_p is the highest frequency component in the sampled signal.

2.3 Quantizing Distortion

$$D_Q \simeq \frac{\Delta^2}{12} \text{ linear quantizing, } \Delta \text{ small}$$

$$\text{SQNR} = \frac{\text{Signal Power}}{D_q}$$

Quantizing distortion for sinusoidal signal, maximum dynamical range used, r bits

$$\text{SQNR} = 1.76 + 6 \cdot r[dB]$$

Quantizing distortion, dynamical range usage expressed in top- and RMS value, r bits

$$\text{SQNR} = 6 \cdot r + 1.76 - 10^{10} \log \left(\frac{A_{peak}}{A_{RMS} \cdot \sqrt{2}} \right)^2 - 10^{10} \log \left(\frac{V}{A_{peak}} \right)^2$$

where $[-V, V]$ is the dynamical range of the quantizer.

2.4 Decimation and Interpolation

Downsampling a factor M

$$\downarrow M \quad y(n) = \{\ldots u(0), u(M), u(2M) \ldots\}$$

$$Y(f) = \tfrac{1}{M} \, \textstyle\sum_{i=0}^{M-1} \; U\left(\tfrac{f-i}{M}\right)$$

Upsampling a factor L

$$\uparrow L \quad w(n) = \{\ldots x(0), \underbrace{0, 0, \ldots,}_{\text{L-1 st}} x(1), \underbrace{0, 0, \ldots,}_{\text{L-1 st}} x(2) \ldots\}$$

$$W(f) = X(fL)$$

3 Analogue Filters

3.1 Filter Approximations if ideal LP filters

General form of the amplitude function of the approximation

$$|H(\Omega)| = \frac{K}{\sqrt{1 + g_N\left(\left(\frac{\Omega}{\Omega_p}\right)^2\right)}} \qquad \Omega = 2\pi F$$

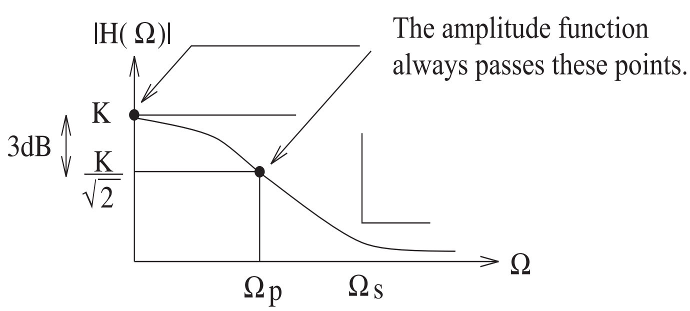
where

$$g_N\left(\left(\frac{\Omega}{\Omega_p}\right)^2\right) \begin{cases} \ll 1 & \left|\frac{\Omega}{\Omega_p}\right| < 1 \\ \gg 1 & \left|\frac{\Omega}{\Omega_p}\right| > 1 \end{cases}$$

and Ω_p is the cut-off frequency of the filter.
Sometimes it is suitable to normalize the angular frequency to Ω_p.
In this section this corresponds to setting $\Omega_p = 1$.

3.1.1 Butterworth Filters

$$|H(\Omega)| = \frac{K}{\sqrt{1 + \left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$

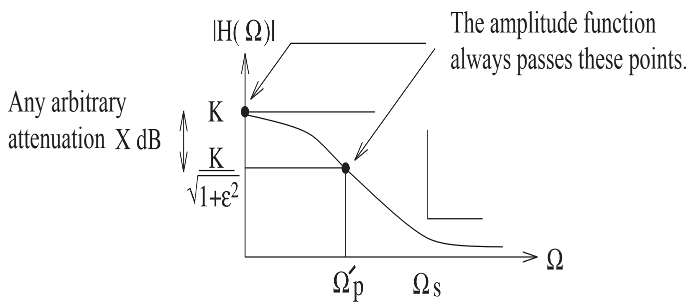


K = The maximum value of the amplitude function.
K = The value of the amplitude functionens for $\Omega = 0$.

Arbitrary attenuation in the pass band (X dB)

$$|H(\Omega)| = \frac{K}{\sqrt{1 + \varepsilon^2\left(\frac{\Omega}{\Omega'_p}\right)^{2N}}}$$

When $\Omega'_p = \Omega_p$ (3dB) then $\varepsilon^2 \approx 1$.



Filter order:

$$N = \frac{\log_{10}\left(\left[\sqrt{\frac{1}{\delta_2^2} - 1}\right] / \varepsilon\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

δ_2 is the maximum allowed value for the amplitude function at the stop band edge:

$$\delta_2 = |H(\Omega_s)|_{\max}$$

The denominator of the system function is a Butterworth polynomial if $\Omega_p = 1$. These polynomials can be found in Table 2.1. For a general Ω_p

$$\mathcal{H}(s) = \frac{K}{\left(\frac{s}{\Omega_p}\right)^N + a_{N-1}\left(\frac{s}{\Omega_p}\right)^{N-1} + \cdots + a_1\left(\frac{s}{\Omega_p}\right) + 1}$$

where $a_1, \ldots, a_{N-1}$ can be found in Table 4.1.

Table 4.1

Coefficients a_ν in Butterworth polynomials $s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + 1$

N	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1							
2	$\sqrt{2}$						
3	2	2					
4	2.613	3.414	2.613				
5	3.236	5.236	5.236	3.236			
6	3.864	7.464	9.141	7.464	3.864		
7	4.494	10.103	14.606	14.606	10.103	4.494	
8	5.126	13.138	21.848	25.691	21.848	13.138	5.126

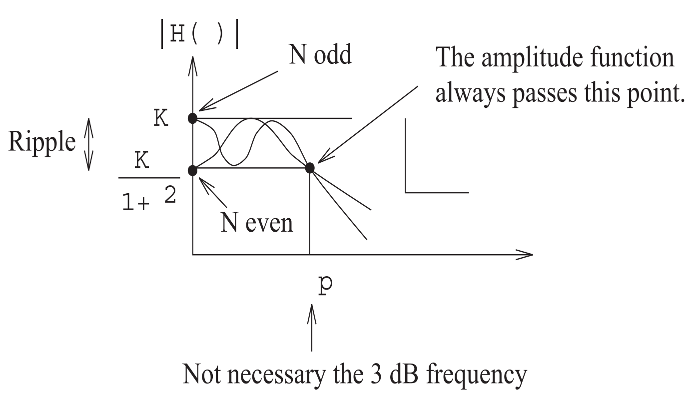
Table 4.2

Factorized Butterworth polynomials for $\Omega_p = 1$. For $\Omega_p \neq 1$ let $s \to s/\Omega_p$.

N	
1	$(s+1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s+1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 1.9318s + 1)$
7	$(s+1)(s^2 + 0.4450s + 1)(s^2 + 1.2465s + 1)(s^2 + 1.8022s + 1)$
8	$(s^2 + 0.3896s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)(s^2 + 1.9622s + 1)$

3.1.2 Chebyshev Filters

$$|H(\Omega)| = \frac{K}{\sqrt{1 + \varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}}$$



Ripple $= 10 \cdot \log(1 + \varepsilon^2)$ dB.
$K =$ The maximum value of the amplitude function.

$K \neq$ the value of the amplitude function for $\Omega = 0$ when N is even.
$T_N(\frac{\Omega}{\Omega_p})$ is a Chebyshev polynomial. (Also denoted $C_N\left(\frac{\Omega}{\Omega_p}\right)$). These can be found in Table 4.3 for $\Omega_p = 1$. For $\Omega_p \neq 1$ let $\Omega \to \frac{\Omega}{\Omega_p}$ in Table 4.3.

System function

$$\mathcal{H}(s) = \frac{K \cdot a_0 \cdot \begin{cases} 1 & N \text{ odd} \\ \frac{1}{\sqrt{1+\varepsilon^2}} & N \text{ even} \end{cases}}{\left(\frac{s}{\Omega_p}\right)^N + a_{N-1}\left(\frac{s}{\Omega_p}\right)^{N-1} + \cdots + a_0}$$

where $\varepsilon, a_0, \ldots, a_{N-1}$ can be found in Table 4.4.
The locations of the poles for $\mathcal{H}(s)$ can be found in Table 4.5 for $\Omega_p = 1$. For $\Omega_p \neq 1$ the pole locations are multiplied with Ω_p.

Table 4.3

Chebyshev polynomials.

$$T_N(\Omega) = \begin{cases} \cos(N \ \arccos \Omega) & |\Omega| \leq 1 \\ \\ \cosh(N \ \text{arccosh}\Omega) & |\Omega| \geq 1 \end{cases} \quad \Omega = 2\pi F$$

or

$$T_N(\Omega) = \frac{\left(\Omega + \sqrt{\Omega^2 - 1}\right)^N + \left(\Omega + \sqrt{\Omega^2 - 1}\right)^{-N}}{2} \quad |\Omega| \geq 1$$

Recursive calculation

$$T_{N+1}(\Omega) = 2\Omega T_N(\Omega) - T_{N-1}(\Omega)$$

Filter order:

$$N = \frac{\text{arccosh}\left(\left[\sqrt{\frac{1}{\delta_2^2} - 1}\right] / \varepsilon\right)}{\text{arccosh}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

δ_2 is the maximum allowed stop band ripple.

Alternative for arccosh:

$$\text{arccosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

N	$T_N(\Omega)$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$
6	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
7	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
8	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
9	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$
10	$512\Omega^{10} - 1280\Omega^8 + 1120\Omega^6 - 400\Omega^4 + 50\Omega^2 - 1$

Table 4.4. Chebyshev filter coefficients a_ν.

0.5dB ripple ($\varepsilon = 0.349,\ \varepsilon^2 = 0.122$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								2.863
2							1.426	1.516
3						1.253	1.535	0.716
4					1.197	1.717	1.025	0.379
5				1.172	1.937	1.309	0.752	0.179
6			1.159	2.172	1.589	1.172	0.432	0.095
7		1.151	2.413	1.869	1.648	0.756	0.282	0.045
8	1.146	2.657	2.149	2.184	1.148	0.573	0.152	0.024

1-dB ripple ($\varepsilon = 0.509,\ \varepsilon^2 = 0.259$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.965
2							1.098	1.102
3						0.989	1.238	0.491
4					0.953	1.454	0.743	0.276
5				0.937	1.689	0.974	0.580	0.123
6			0.928	1.931	1.202	0.939	0.307	0.069
7		0.923	2.176	1.429	1.357	0.549	0.214	0.031
8	0.920	2.423	1.655	1.837	0.447	0.448	0.107	0.017

2-dB ripple ($\varepsilon = 0.765,\ \varepsilon^2 = 0.585$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.307
2							0.804	0.823
3						0.738	1.022	0.327
4					0.716	1.256	0.517	0.206
5				0.705	1.499	0.693	0.459	0.082
6			0.701	1.745	0.867	0.771	0.210	0.051
7		0.698	1.994	1.039	1.144	0.383	0.166	0.020
8	0.696	2.242	1.212	1.579	0.598	0.359	0.073	0.013

3-dB*) ripple ($\varepsilon = 0.998,\ \varepsilon^2 = 0.995$).

N	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
1								1.002
2							0.645	0.708
3						0.597	0.928	0.251
4					0.581	1.169	0.405	0.177
5				0.575	1.415	0.549	0.408	0.063
6			0.571	1.663	0.691	0.699	0.163	0.044
7		0.568	1.911	0.831	1.052	0.300	0.146	0.016
8	0.567	2.161	0.972	1.467	0.472	0.321	0.056	0.011

*) The table was calculated with "exactly" 3dB, not with $20 \cdot \log \sqrt{2} \approx 3.01$dB. Hence $\varepsilon \neq 1$ and $a_0 \neq 1$ for $N = 1$.

Table 4.5. Pole locations for Chebyshev filters.

0.5dB ripple ($\varepsilon = 0.349,\ \varepsilon^2 = 0.122$).

$N = 1$	2	3	4	5	6	7	8
-2.863	-0.713	-0.626	-0.175	-0.362	-0.078	-0.256	-0.044
	±j1.004		±j1.016		±j1.008		±j1.005
		-0.313	-0.423	-0.112	-0.212	-0.057	-0.124
		±j1.022	±j0.421	±j1.011	±j0.738	±j1.006	±j0.852
				-0.293	-0.290	±0.160	-0.186
				±j0.625	±j0.270	±j0.807	±j0.570
						-0.231	-0.220
						±j0.448	±j0.200

1-dB ripple ($\varepsilon = 0.509,\ \varepsilon^2 = 0.259$).

$N = 1$	2	3	4	5	6	7	8
-1.965	-0.549	-0.494	-0.139	-0.289	-0.062	-0.205	-0.035
	±j0.895		±j0.983		±j0.993		±j0.996
		-0.247	-0.337	-0.089	-0.170	-0.046	-0.100
		±j0.966	±j0.407	±j0.990	±j0.727	±j0.995	±j0.845
				-0.234	-0.232	-0.128	-0.149
				±j0.612	±j0.266	±j0.798	±j0.564
						-0.185	-0.176
						±j0.443	±j0.198

2-dB ripple ($\varepsilon = 0.765,\ \varepsilon^2 = 0.585$).

$N = 1$	2	3	4	5	6	7	8
-1.307	-0.402	-0.369	-0.105	-0.218	-0.047	-0.155	-0.026
	±j0.813		±j0.958		±j0.982		±j0.990
		-0.184	-0.253	-0.067	-0.128	-0.034	-0.075
		±j0.923	±0.397	±j0.973	±0.719	±j0.987	±j0.839
				-0.177	-0.175	-0.097	-0.113
				±j0.602	±j0.263	±j0.791	±j0.561
						-0.140	-0.133
						±j0.439	±j0.197

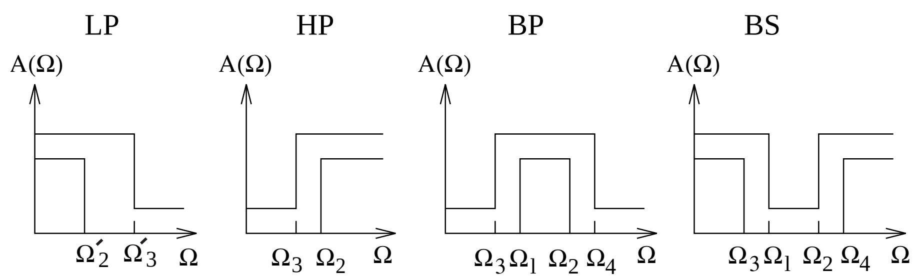
3-dB*) ripple ($\varepsilon = 0.998,\ \varepsilon^2 = 0.995$).

$N = 1$	2	3	4	5	6	7	8
-1.002	-0.322	-0.299	-0.085	-0.177	-0.038	-0.126	-0.021
	±j0.777		±j0.946		±j0.976		±0.987
		-0.1493	-0.206	-0.055	-0.104	-0.028	-0.061
		±j0.904	±j0.392	±j0.966	±0.715	±j0.983	±j0.836
				-0.144	-0.143	-0.079	-0.092
				±j0.597	±j0.262	±j0.789	±j0.559
						-0.114	-0.108
						±j0.437	±j0.196

*) See note in Table 4.4.

3.2 Frequency Transformations of Analogue Filter

1. Start out from the frequencies in the filter specification in the analogue high pass-, band pass- or band stop filter.
2. Transform to the frequencies of the LP filter Ω_2' and Ω_3'.
3. Find the coefficients of the LP filter coefficients.
4. Transform back to the original filter (HP, BP, BS) by replacing s i the low pass filter $H(s)$; see below.



HP ⇒ LP $\quad \Omega_2' = \frac{1}{\Omega_2} \qquad \Omega_3' = \frac{1}{\Omega_3}$

BP ⇒ LP $\quad \Omega_2' = \Omega_2 - \Omega_1 \qquad \Omega_3' = \Omega_4 - \Omega_3 \qquad \Omega_1\Omega_2 = \Omega_3\Omega_4 = \Omega_I^2$

BS ⇒ LP $\quad \Omega_2' = \frac{\Omega_I^2}{\Omega_4 - \Omega_3} \qquad \Omega_3' = \frac{\Omega_I^2}{\Omega_2 - \Omega_1} \qquad \Omega_1\Omega_2 = \Omega_3\Omega_4 = \Omega_I^2$

LP		HP	BP	BS
s	$\rightarrow$	$\frac{1}{s}$	$s + \frac{\Omega_I^2}{s}$	$\frac{\Omega_I^2}{s + \frac{\Omega_I^2}{s}}$

4 Discrete Time Filters

4.1 FIR Filters and IIR Filters

$$\begin{aligned}
\mathcal{H}(z) &= b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{-M+1} \\
h(n) &= \begin{cases} b_n & 0 \leq n \leq M-1 \\ 0 & \text{Else} \end{cases}
\end{aligned}$$

Linear phase FIR Filter :
Symmetrical impulse response $h(n) = h(M-1-n)$
Anti-symmetrical impulse response $h(n) = -h(M-1-n)$

IIR Filter

$$\begin{aligned}
\mathcal{H}(z) &= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \\
h(n) &= Z^{-1}\{\mathcal{H}(z)\}
\end{aligned}$$

4.2 Construction of FIR Filter

4.2.1 FIR Filter using the window method

Impulse response

$$h(n) = h_d(n) \cdot w(n)$$

with desired impulse response $h_d(n)$ and time window $w(n)$.
Filters designed using the window method have linear phase.

Desired impulse response $h_d(n)$ defined for $0 \leq n \leq M-1$
Odd filter length

Low pass:

$$\begin{aligned}
h_d(n) &= \frac{\omega_c}{\pi} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\omega_c \left(n - \frac{M-1}{2}\right)} \\
h_d(\frac{M-1}{2}) &= \frac{\omega_c}{\pi} \\
H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & 0 \leq |\omega| < \omega_c \\ 0 & \text{Else} \end{cases}
\end{aligned}$$

High pass:

$$\begin{aligned}
h_d(n) &= \delta\left(n - \frac{M-1}{2}\right) - \frac{\omega_c}{\pi} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\omega_c \left(n - \frac{M-1}{2}\right)} \\
H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & \omega_c < |\omega| < \pi \\ 0 & \text{Else} \end{cases}
\end{aligned}$$

Band pass:

$$\begin{aligned} h_d(n) &= 2\cos\left(\omega_0\left(n-\frac{M-1}{2}\right)\right)\cdot\frac{\omega_c}{\pi}\,\frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\omega_c\left(n-\frac{M-1}{2}\right)} \\ H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & \omega_0-\omega_c<|\omega|<\omega_0+\omega_c \\ 0 & \text{Else} \end{cases} \end{aligned}$$

Band stop:

$$\begin{aligned} h_d(n) &= \delta\left(n-\frac{M-1}{2}\right)-2\cos\left(\omega_0\left(n-\frac{M-1}{2}\right)\right)\cdot\frac{\omega_c}{\pi}\,\frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\omega_c\left(n-\frac{M-1}{2}\right)} \\ H_d(\omega) &= \begin{cases} e^{-j\omega\ (M-1)/2} & 0<|\omega|<\omega_0-\omega_c \text{ and } \omega_0+\omega_c<|\omega|<\pi \\ 0 & \text{Else} \end{cases} \end{aligned}$$

The spectrum of the filter $H(\omega) = H_d(\omega) * W(\omega)$.
At the cut-off frequency ω_c the attenuation is 6dB.
When dimensioning filters, the tables below offers a rough estimation of the required filter length M.

Table 5.1

The main and sidelobe size for some common window functions.

Window	Approximate main lobe width (rad)	Largest sidelobe relative main lobe (dB)
Rektangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-27
Hanning	$8\pi/M$	-32
Hamming	$8\pi/M$	-43
Blackman	$12\pi/M$	-58

Table 5.2

Sidelobe size for some filters designed using the window method.

Window	Approximate largest sidelobe (dB)
Rektangulr	-20
Bartlett	-27
Hanning	-40
Hamming	-50
Blackman	-70
Kaiser ($\beta = 4.54$)	-50
Kaiser ($\beta = 6.76$)	-70
Kaiser ($\beta = 8.96$)	-90

Window functions defined for $0 \leq n \leq M-1$ (Odd filter length M)

Rectangular window

$$w(n) = 1$$

Bartlett (Triangular window)

$$w(n) = 1 - \frac{\left|n - \frac{M-1}{2}\right|}{\frac{M-1}{2}}$$

Hanning window

$$\begin{aligned} w(n) &= 0.5 \left(1 + \cos \frac{2\pi \left(n - \frac{M-1}{2}\right)}{M-1}\right) = \\ &= 0.5 \left(1 - \cos \frac{2\pi n}{M-1}\right) \end{aligned}$$

Hamming window

$$\begin{aligned} w(n) &= 0.54 + 0.46 \ \cos \ \frac{2\pi \left(n - \frac{M-1}{2}\right)}{M-1} = \\ &= 0.54 - 0.46 \ \cos \ \frac{2\pi n}{M-1} \end{aligned}$$

Blackman window

$$\begin{aligned} w(n) &= 0.42 + 0.5 \ cos \ \frac{2\pi \left(n - \frac{M-1}{2}\right)}{M-1} + \\ & +0.08 \ cos \ \frac{4\pi \left(n - \frac{M-1}{2}\right)}{M-1} = \\ &= 0.42 - 0.5 \ \cos \ \frac{2\pi n}{M-1} + 0.08 \ \cos \ \frac{4\pi n}{M-1} \end{aligned}$$

Kaiser window

$$w(n) = \frac{I_0 \left(\beta \sqrt{1 - [2(n - \frac{M-1}{2})/(M-1)]^2}\right)}{I_0(\beta)}$$

$I_0(x)$ is a Bessel function, whose value can be calculated according to

$$I_0(x) = 1 + \sum_{k=1}^{L} \left[\frac{(x/2)^k}{k!}\right]^2$$

Usually $L < 25$.
The value of β is decided from the stop band attenuation D.

Stop band attenuation D	β
$D \leq 21\text{dB}$	0
$21\text{dB} < D < 50\text{dB}$	$0,5842(D-21)^{0,4} + 0,07886(D-21)$
$D \geq 50\text{dB}$	$0,1102(D-8,7)$

Filter length M:

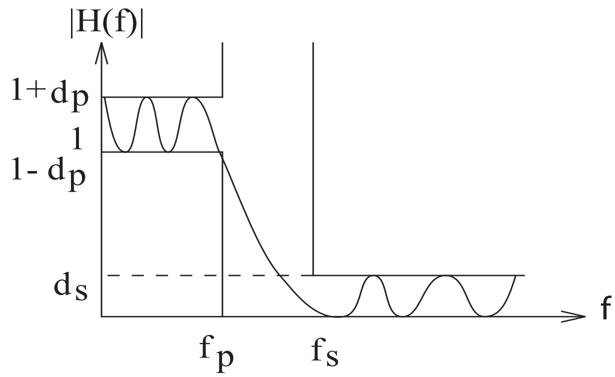
$$M \geq \frac{D - 7,95}{14,36\Delta f}$$

$$\Delta f = f_s - f_p$$

f_p Pass band frequency (normalized frequency)
f_s Stop band frequency (normalized frequency)

4.2.2 Equiripple FIR Filter

Dimensioning of equiripple filters when using the Remez algorithm. Approximately according to Kaiser.



$$
\begin{aligned}
\text{Filter length } M &= \frac{-20\log_{10}\left(\sqrt{\delta_p\delta_s}\right)-13}{14.6\Delta f}+1 \\
\Delta f &= f_s - f_p
\end{aligned}
$$

4.2.3 FIR Filters using Least-Squares

Minimizing

$$\mathcal{E} = \sum_n [x(n) * h(n) - d(n)]^2$$

yields

$$\sum_{n=0}^{M-1} h(n) r_{xx}(n-\ell) = r_{dx}(\ell) \quad \ell = 0, \ldots, M-1$$

and

$$\mathcal{E}_{\min} = r_{dd}(0) - \sum_{k=0}^{M-1} h(k) r_{dx}(k)$$

where $r_{xx}(\ell)$ is the correlation function for $x(n)$ and $r_{dx}(\ell)$ is the cross correlation between $d(n)$ and $x(n)$.
This can be written on matrix form as

$$
\begin{aligned}
&\mathbf{R}_{xx} \cdot \mathbf{h} = \mathbf{r}_{dx} \\
&\mathbf{h} = \mathbf{R}_{xx}^{-1} \cdot \mathbf{r}_{dx} \\
&\mathcal{E}_{\min} = r_{dd}(0) - \mathbf{h}^T \cdot \mathbf{r}_{dx}
\end{aligned}
$$

4.3 Constructing IIR Filters starting from Analogue Filters

4.3.1 The Impulse-Invariant Method

$$h(n) = h_a(t)|_{t=nT} = h_a(nT)$$

$$H(z) = \sum_{n=0}^{\infty} h_a(nT) z^{-n}$$

1.

$$h_a(t) = e^{-\sigma_0 t} \longleftrightarrow \mathcal{H}_a(s) = \frac{1}{s+\sigma_0}$$

$$\Rightarrow \mathcal{H}(z) = \frac{1}{1 - e^{-\sigma_0 T} z^{-1}}$$

2.

$$h_a(t) = e^{-\sigma_0 t}\cos\Omega_0 t \longleftrightarrow \mathcal{H}_a(s) = \frac{s+\sigma_0}{(s+\sigma_0)^2 + \Omega_0^2}$$

$$\Rightarrow \mathcal{H}(z) = \frac{1 - z^{-1}e^{-\sigma_0 T}\cos\Omega_0 T}{1 - 2z^{-1}e^{-\sigma_0 T}\cos\Omega_0 T + z^{-2}e^{-2\sigma_0 T}}$$

3.

$$h_a(t) = e^{-\sigma_0 t}\sin\Omega_0 t \longleftrightarrow \mathcal{H}_a(s) = \frac{\Omega_0}{(s+\sigma_0)^2 + \Omega_0^2}$$

$$\Rightarrow \mathcal{H}(z) = \frac{z^{-1}e^{-\sigma_0 T}\sin\Omega_0 T}{1 - 2z^{-1}e^{-\sigma_0 T}\cos\Omega_0 T + z^{-2}e^{-2\sigma_0 T}}$$

4.3.2 Bilinear Transformation

Frequency transformation ("prewarp")

$$\Omega_{\text{prewarp}} = \frac{2}{T}\ \tan\frac{\omega}{2}$$

Analogue construction of filters in the variable $\Omega_{prewarp} = 2\pi F_{\text{prewarp}}$ $(\ \omega = 2\pi f)$
.

$$\mathcal{H}(z) = \mathcal{H}_a(s) \text{ where } s = \frac{2}{T}\ \frac{1-z^{-1}}{1+z^{-1}}$$

T is a normalization factor (can often be set to 1).

4.3.3 Quantizing of Coefficients

Change in pole locations when the coefficients $a_1, \ldots, a_k$ is changed $\Delta a_1, \ldots, \Delta a_k$

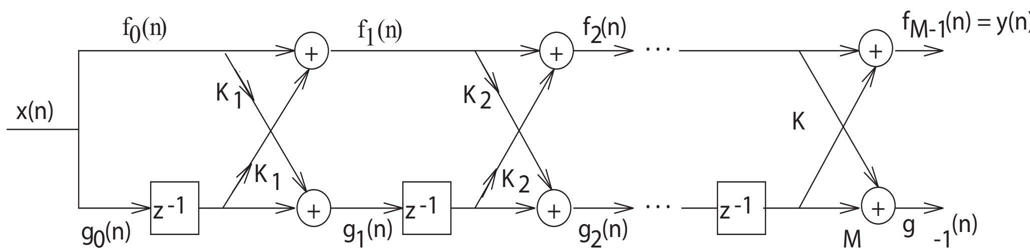
$$\Delta p_i \approx \frac{\partial p_i}{\partial a_1}\ \Delta a_1 + \cdots + \frac{\partial p_i}{\partial a_k}\ \Delta a_k$$

If using normal form (Direct form II) then

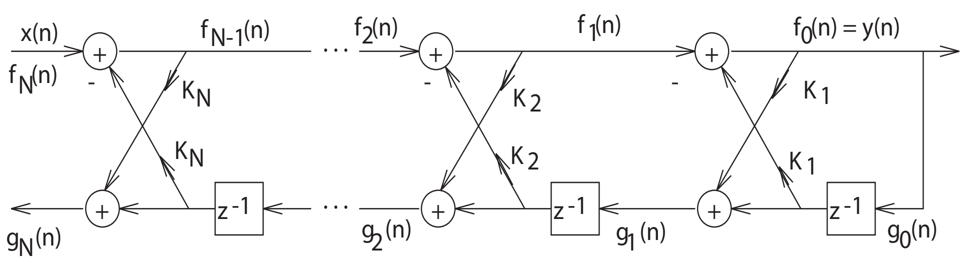
$$\frac{\partial p_i}{\partial a_j} = \underbrace{\frac{-p_i^{k-j}}{(p_i - p_1)(p_i - p_2)\ldots(p_i - p_k)}}_{k-1 \text{ factors}}$$

$(p_i - p_i)$ should not be included

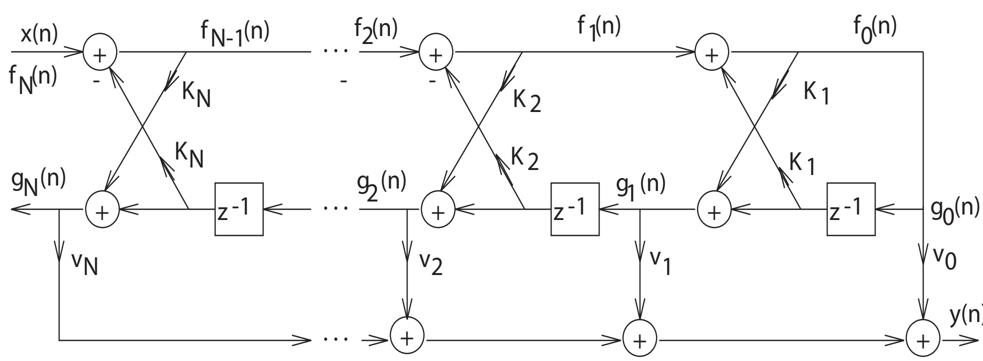
4.4 Lattice Filters



Lattice FIR



Lattice all-pole IIR



Lattice-ladder

Conversion from Lattice to Direct form:

$$A_0(z) = B_0(z) = 1$$

$$\begin{cases} A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \\ B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z) \end{cases}$$

$$m = 1, 2, \ldots, M-1$$

Relationship between $A_m(z)$ and $B_m(z)$

$$A_m(z) = \alpha_m(0) + \alpha_m(1)z^{-1} + ... + \alpha_m(m-1)z^{-m+1} + \alpha_m(m)z^{-m}$$

$$B_m(z) = \beta_m(0) + \beta_m(1)z^{-1} + ... + \beta_m(m-1)z^{-m+1} + \beta_m(m)z^{-m}$$

$$\begin{aligned} B_m(z) &= z^{-m} A_m(z^{-1}) \\ \beta_m(k) &= \alpha_m(m-k) \\ k &= 0, 1, \ldots, m \end{aligned}$$

Conversion from Direct form to Lattice:

$$A_{m-1}(z) = \frac{1}{1-K_m^2} \left(A_m(z) - K_m B_m(z)\right)$$

$$m = M-1, M-2, \ldots, 1$$

Reflection coefficient:

$$K_m = \alpha_m(m)$$

FIR Filter

$$H(z) = A_N(z)$$

$$A_{M-1}(z) = A_N(z)$$

IIR Filter (all-pole)

$$H(z) = \frac{1}{A_N(z)}$$

$$A_{M-1}(z) = A_N(z)$$

Lattice-Ladder

Conversion from Direct form to Lattice-Ladder:

$$H(z) = \frac{C_M(z)}{A_N(z)} = \frac{c_0 + c_1 z^{-1} \ldots c_M z^{-M}}{A_N(z)}$$

$$C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

$$v_m = c_m(m) \quad m = 0, 1, \ldots, M$$

5 Correlation

Correlation, Cross Correlation, spectrum, cross spectrum and coherence between input and output signal.

Continuous time process:

$$
\begin{aligned}
y(t) &= h(t) * x(t) \\
Y(F) &= H(F) \cdot X(F) \\
r_{yy}(\tau) &= r_{hh}(\tau) * r_{xx}(\tau) \\
R_{yy}(F) &= |H(F)|^2 R_{xx}(F) \\
r_{yx}(\tau) &= h(\tau) * r_{xx}(\tau) \\
R_{yx}(F) &= H(F) \cdot R_{xx}(F) \\
r_{xx}(\tau) &= \int_t x(t)x(t-\tau)dt \\
r_{yx}(\tau) &= \int_t y(t)x(t-\tau)dt \\
\gamma_{xx}(\tau) &= E\{x(t)x(t-\tau)\} \\
\gamma_{yx}(\tau) &= E\{y(t)x(t-\tau)\}
\end{aligned}
$$

Discrete time process:

$$
\begin{aligned}
y(n) &= h(n) * x(n) \\
Y(f) &= H(f) \cdot X(f) \\
r_{yy}(n) &= r_{hh}(n) * r_{xx}(n) \\
R_{yy}(f) &= |H(f)|^2 \cdot R_{xx}(f) \\
r_{yx}(n) &= h(n) * r_{xx}(n) \\
R_{yx}(f) &= H(f) \cdot R_{xx}(f) \\
r_{xx}(n) &= \sum_\ell x(\ell)x(\ell-n) \\
r_{yx}(n) &= \sum_\ell y(\ell)x(\ell-n) \\
\gamma_{xx}(n) &= E\{x(\ell)x(\ell-n)\} \\
\gamma_{yx}(n) &= E\{y(\ell)x(\ell-n)\}
\end{aligned}
$$

Normally distributed stochastic variables. $X_i \in N(m_i, \sigma_i)$

$$
\begin{aligned}
E\{X_1X_2X_3X_4\} &= E\{X_1X_2\}\ E\{X_3X_4\} + E\{X_1X_3\}\ E\{X_2X_4\} + \\
&+ E\{X_1X_4\}\ E\{X_2X_3\} - 2m_1m_2m_3m_4
\end{aligned}
$$

6 Spectrum Estimation

Spectrum Estimation

$$\gamma_{xx}(m) = E\{x(n)x(n+m)\} \text{ autocorrelation}$$

$$\Gamma_{xx}(f) = \sum_{m=-\infty}^{\infty} \gamma_{xx} e^{-j2\pi fm} \text{ power spectrum}$$

$$r_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-m-1} x(n)x(n+m) \quad 0 \le m \le N-1 \quad \text{auto correlation (estimate)}$$

$$P_{xx}(f) = \sum_{m=-N+1}^{N-1} r_{xx}(m)e^{-j2\pi fm} = \frac{1}{N}\left|\sum_{m=0}^{N-1} x(m)e^{-j2\pi fm}\right|^2 \quad \text{power spectrum (estimate)}$$

Periodogram

$$P_{xx}(f) = \frac{1}{N}\left|\sum_{m=0}^{N-1} x(m)e^{-j2\pi fm}\right|^2 \quad \text{power spectrum (estimate)}$$

$$E\{r_{xx}(m)\} = \left(1 - \frac{|m|}{N}\right)\gamma_{xx}(m) \to \gamma_{xx}(m) \text{ when } N \to \infty$$

$$var(r_{xx}(m)) \approx \frac{1}{N}\sum_{n=-\infty}^{\infty}[\gamma_{xx}^2(n) + \gamma_{xx}(n-m)\gamma_{xx}(n+m)] \to 0 \text{ when } N \to \infty$$

$$E\{P_{xx}(f)\} = \int_{-1/2}^{1/2} \Gamma_{xx}(\alpha)W_B(f-\alpha)d\alpha$$

where $W_B(f)$ is the Fourier transform of the Bartlett window $\left(1 - \frac{|m|}{N}\right)$

$$var(P_{xx}(f)) = \Gamma_{xx}^2(f)\left[1 + \left(\frac{sin2\pi fN}{Nsin2\pi f}\right)^2\right] \to \Gamma_{xx}^2(f) \text{ when } N \to \infty$$

if $x(n)$ Gaussian.

Periodogram using DFT:

$$P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{nk}{N}}\right|^2 \quad k = 0, \ldots, N-1$$

Bartletts method

Averaging periodograms.
Divide the N sample data sequence $x(n)$ into K blocks with M samples in each block. The blocks are not overlapping ($N = K \cdot M$).

$$x_i(n) = x(n + iM) \quad i = 0, 1 \ldots, K-1 \quad \text{and } n = 0, 1 \ldots, M-1$$

$$\begin{aligned} P_{xx}^i(f) &= \frac{1}{M} \left| \sum_{m=0}^{N-1} x_i(m) e^{-j2\pi f m} \right|^2 \quad i = 0, 1 \ldots, K-1 \\ P_{xx}^B(f) &= \frac{1}{K} \sum_{i=0}^{K-1} P_{xx}^i(f) \end{aligned}$$

$$\begin{aligned} E\{P_{xx}^B(f)\} &= \frac{1}{K} \sum_{i=0}^{K-1} E\{P_{xx}^i(f)\} = E\{P_{xx}^i(f)\} \\ E\{P_{xx}^i(f)\} &= \frac{1}{M} \int_{-1/2}^{1/2} \Gamma_{xx}(\alpha) W_B(f-\alpha) d\alpha \end{aligned}$$

where $W_B(f)$ is the Fourier Transform of the Bartlett window $\left(1 - \frac{|m|}{N}\right)$

$$W_B(f) = \frac{1}{M} \left(\frac{\sin \pi f M}{\sin \pi f} \right)^2$$

$$\begin{aligned} \text{var}(P_{xx}^B(f)) &= \frac{1}{K^2} \sum_{i=0}^{K-1} \text{var}(P_{xx}^i(f))\} \\ \text{var}(P_{xx}^B(f)) &= \frac{1}{K} \Gamma_{xx}^2(f) \left[1 + \left(\frac{sin 2\pi f N}{N sin 2\pi f} \right)^2 \right] \end{aligned}$$

Welch's method

Averaging modified periodograms.
Divide the N sample data sequence $x(n)$ into L block with M samples in each block. The blocks may be overlapping. The starting point for block i is given by iD. $D = M$ means no block overlap and $D = M/2$ yields 50% overlap.

$$x_i(n) = x(n + iD) \quad i = 0, 1 \ldots, L-1 \quad \text{and } n = 0, 1 \ldots, M-1$$

Modified periodogram.

$$
\begin{aligned}
\tilde{P}^i_{xx}(f) &= \frac{1}{MU}\left|\sum_{m=0}^{N-1} x_i(m)w(n)e^{-j2\pi fm}\right|^2 \quad i = 0, 1 \ldots, L-1 \\
w(n) & \quad \text{window function} \\
U &= \frac{1}{M}\sum_{n=0}^{N-1} w^2(n) \\
P^W_{xx}(f) &= \frac{1}{L}\sum_{i=0}^{L-1} \tilde{P}^i_{xx}(f)
\end{aligned}
$$

$$
\begin{aligned}
E\{P^W_{xx}(f)\} &= \frac{1}{L}\sum_{i=0}^{L-1} E\{\tilde{P}^i_{xx}(f)\} = E\{\tilde{P}^i_{xx}(f)\} \\
E\{\tilde{P}^i_{xx}(f)\} &= \int_{-1/2}^{1/2} \Gamma_{xx}(\alpha)W(f-\alpha)d\alpha
\end{aligned}
$$

$$
W(f) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} w(n)e^{-j2\pi fn}\right|^2
$$

No block overlap ($L = K$)

$$
\begin{aligned}
\text{v}ar(P^W_{xx}(f)) &= \frac{1}{L}\text{v}ar(\tilde{P}^i_{xx}(f))\} \\
&\approx \frac{1}{L}\Gamma^2_{xx}(f)
\end{aligned}
$$

50% block overlap ($L = 2K$) and Bartlett window (triangular window).

$$
\text{v}ar(P^W_{xx}(f)) \approx \frac{9}{8L}\Gamma^2_{xx}(f)
$$

Averaging Periodograms

Quality factor

$$Q = \frac{[E\{P_{xx}(f)\}]^2}{var(P_{xx}(f))}$$

Relative variance $\frac{1}{Q}$
$Q \approx$ time-bandwidth product.

Periodogram	$\Delta f = \frac{0.9}{M}$	$Q = 1$	
Bartlett $(N = K \cdot M)$	$\Delta f = \frac{0.9}{M}$	$Q = \frac{N}{M}$	Rectangular window No overlap
Welch $(N = L \cdot M)$	$\Delta f = \frac{1.28}{M}$	$Q = \frac{16}{9} \cdot \frac{N}{M}$	Triangular window 50% overlap
Welch $(N = L \cdot M)$	$\Delta f = \frac{1.50}{M}$	$Q = \frac{3}{1.08 \cdot 2} \cdot \frac{N}{M}$	Hanning window 50% overlap
Welch	$\Delta f = \frac{1.50}{M}$	$Q = \frac{3}{2} \cdot \frac{N}{M}$	Hanning window 62,5% overlap
Blackman/Tukey	$\Delta f = \frac{0.6}{M}$	$Q = \frac{1}{2} \cdot \frac{N}{M}$	Rectangular window
	$\Delta f = \frac{0.9}{M}$	$Q = \frac{3}{2} \cdot \frac{N}{M}$	Triangular window

The resolution Δf is calculated at the -3dB points from the main lobe of the window.

A Basic Relationships

A.1 Trigonometrical formulas

$$\begin{aligned}
&\sin\alpha = \cos(\alpha - \pi/2)\\
&\cos\alpha = \sin(\alpha + \pi/2)\\
&\cos^2\alpha + \sin^2\alpha = 1\\
&\cos^2\alpha - \sin^2\alpha = \cos 2\alpha\\
&2\sin\alpha\cos\alpha = \sin 2\alpha\\
&\sin(-\alpha) = -\sin\alpha\\
&\cos(-\alpha) = \cos\alpha\\
&\cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)
\end{aligned}$$

$$\begin{aligned}
&\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\\
&\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\\
&2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)\\
&2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)\\
&2\cos\alpha\cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)\\
&\sin\alpha + \sin\beta = 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}\\
&\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}
\end{aligned}$$

$$\cos\alpha = \tfrac{1}{2}\,(e^{j\alpha} + e^{-j\alpha}), \quad \sin\alpha = \tfrac{1}{2j}\,(e^{j\alpha} - e^{-j\alpha}), \quad e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$A\cos\alpha + B\sin\alpha = \sqrt{A^2 + B^2}\cos(\alpha - \beta)$$

where $\cos\beta = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\beta = \frac{B}{\sqrt{A^2+B^2}}$

$$\text{and } \beta = \begin{cases} \arctan\frac{B}{A} & \text{if } A \geq 0 \\ \arctan\frac{B}{A} + \pi & \text{if } A < 0 \end{cases}$$

$$A\cos\alpha + B\sin\alpha = \sqrt{A^2 + B^2}\sin(\alpha + \beta)$$

where $\cos\beta = \frac{B}{\sqrt{A^2+B^2}}$, $\sin\beta = \frac{A}{\sqrt{A^2+B^2}}$

$$\text{and } \beta = \begin{cases} \arctan\frac{A}{B} & \text{if } B \geq 0 \\ \arctan\frac{A}{B} + \pi & \text{if } B < 0 \end{cases}$$

Degrees	Rad	sin	cos	tan	cot
0	0	0	1	0	$\pm\infty$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	$\pm\infty$	0

A.2 Some Common Relationships

Sum of geometrical series

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N & \text{if } a = 1 \\ \frac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

Summation of sinusoid over an even number of periods

$$\sum_{n=0}^{N-1} e^{j2\pi\ kn/N} = \begin{cases} N & \text{if } k = 0, \pm N, \dots \\ 0 & \text{Else} \end{cases}$$

A.3 Matrix Theory

Matrix notation A and vector x

A matrix $\mathbf{A}$ with dimension $m \times n$ and a vector $\mathbf{x}$ with dimension n is defined by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The matrix $\mathbf{A}$ is symmetrical if $a_{ij} = a_{ji}\ \forall\ ij$.
$\mathbf{I}$ denotes the unity matrix.

Transposing matrix A

$$\mathbf{B} = \mathbf{A}^T \text{ where } b_{ij} = a_{ji}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Matrix A determinant

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det \mathbf{M}_{ij}$$

where $\mathbf{M}_{ij}$ is the resulting matrix if row i and column j in matrix $\mathbf{A}$ is deleted.

$$\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$$

Especially, for a 2×2 matrix:

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$$

Inverse of matrix A

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \quad (\text{if } \det \mathbf{A} \# 0)$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \mathbf{C}^T$$

where $\mathbf{C}$ is defined by

$$c_{ij} = (-1)^{i+j} \cdot \det \mathbf{M}_{ij}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Especially for a 2×2 matrix:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Eigenvalues and Eigenvectors

The eigenvalues ($\lambda_i,\ i = 1, 2, \dots, n$) and the eigenvectors ($\mathbf{q}_i,\ i = 1, 2, \dots, n$) are solutions for the equation system

$$\mathbf{Aq} = \lambda\mathbf{q} \text{ or } (\mathbf{A} - \lambda\mathbf{I})\mathbf{q} = 0$$

The eigenvalues can be calculated as solutions to the characteristic equation (secular equation) for $\mathbf{A}$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_0 = 0$$

$\det(\lambda\mathbf{I} - \mathbf{A})$ is called the characteristic polynomial (the secular polynomial) for $\mathbf{A}$.

B Transforms

B.1 The Laplace Transform

B.1.1 The Laplace transform of causal signals

In the table below $f(t) = 0$ for $t < 0$ (i.e. $f(t) \cdot u(t) = f(t)$).

1. $f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}(s) e^{st} ds$ $\longleftrightarrow$ $\mathcal{F}(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$
2. $\sum_{\nu} a_{\nu} f_{\nu}(t)$ $\longleftrightarrow$ $\sum_{\nu} a_{\nu} \mathcal{F}_{\nu}(s)$ Linearity
3. $f(at)$ $\longleftrightarrow$ $\frac{1}{a}\ \mathcal{F}(\frac{s}{a})$ Scaling
4. $\frac{1}{a}\ f(\frac{t}{a})$ $\longleftrightarrow$ $\mathcal{F}(as)\ a > 0$ Scaling
5. $f(t - t_0);\ t \geq t_0$ $\longleftrightarrow$ $\mathcal{F}(s)\ e^{-st_0}$ Time shift
6. $f(t) \cdot e^{-at}$ $\longleftrightarrow$ $\mathcal{F}(s + a)$ Frequency shift
7. $\frac{d^n f}{dt^n}$ $\longleftrightarrow$ $s^n \mathcal{F}(s)$ Derivation
8. $\int_{0-}^{t} f(\tau) d\tau$ $\longleftrightarrow$ $\frac{1}{s}\ \mathcal{F}(s)$ Integration
9. $(-t)^n f(t)$ $\longleftrightarrow$ $\frac{d^n \mathcal{F}(s)}{ds^n}$ Derivation in frequency domain
10. $\frac{f(t)}{t}$ $\longleftrightarrow$ $\int_{s}^{\infty} \mathcal{F}(z) dz$ Integration in frequency domain
11. $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s \cdot \mathcal{F}(s)$ Initial value theorem
12. $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot \mathcal{F}(s)$ End value theorem
13. $f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_1(t - \tau) f_2(\tau) d\tau$ $\longleftrightarrow$ $\mathcal{F}_1(s) \cdot \mathcal{F}_2(s)$ Convolution in time domain
14. $f_1(t) \cdot f_2(t)$ $\longleftrightarrow$ $\frac{1}{2\pi j}\ \mathcal{F}_1(s) * \mathcal{F}_2(s) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}_1(z) \cdot \mathcal{F}_2(s - z) \cdot dz$ Convolution in frequency domain
15. $\int_{0-}^{\infty} f_1(t) \cdot f_2(t) dt = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathcal{F}_1(s) \cdot \mathcal{F}_2(-s) ds$ Parseval's relation

16. $\delta(t) \longleftrightarrow 1$

17. $\delta^n(t) \longleftrightarrow s^n$

18. $1 \longleftrightarrow \frac{1}{s}$

19. $\frac{1}{n!}\, t^n \longleftrightarrow \frac{1}{s^{n+1}}$

20. $e^{-\sigma_0 t} \longleftrightarrow \frac{1}{s+\sigma_0}$

21. $\frac{1}{(n-1)!}\, t^{n-1} e^{-\sigma_0 t} \longleftrightarrow \frac{1}{(s+\sigma_0)^n}$

22. $\sin \Omega_0 t \longleftrightarrow \frac{\Omega_0}{s^2+\Omega_0^2}$

23. $\cos \Omega_0 t \longleftrightarrow \frac{s}{s^2+\Omega_0^2}$

24. $t \cdot \sin \Omega_0 t \longleftrightarrow \frac{2\Omega_0 s}{(s^2+\Omega_0^2)^2}$

25. $t \cdot \cos \Omega_0 t \longleftrightarrow \frac{s^2-\Omega_0^2}{(s^2+\Omega_0^2)^2}$

26. $e^{-\sigma_0 t} \sin \Omega_0 t \longleftrightarrow \frac{\Omega_0}{(s+\sigma_0)^2+\Omega_0^2}$

27. $e^{-\sigma_0 t} \cos \Omega_0 t \longleftrightarrow \frac{s+\sigma_0}{(s+\sigma_0)^2+\Omega_0^2}$

28. $e^{-\sigma_0 t} \sin(\Omega_0 t + \phi) \longleftrightarrow \frac{(s+\sigma_0)\sin\phi + \Omega_0 \cos\phi}{(s+\sigma_0)^2+\Omega_0^2}$

B.1.2 One-sided Laplace transform of non-causal signals

Notation

$\mathcal{F}^+(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$ — Single sided Laplace transform, $f(t)$ not necessarily causal.

$\mathcal{F}(s) = \mathcal{F}^+(s)$ — For causal signals

Taking the derivative of $f(t)$ yields

$$\frac{d}{dt}\, f(t) \longleftrightarrow s \cdot \mathcal{F}^+(s) - f(0-) \qquad \text{Single derivation}$$

$$\frac{d^n}{dt}\, f(t) \longleftrightarrow s^n \mathcal{F}^+(s) - s^{n-1} f(0-)$$
$$-s^{n-2} f^{(1)}(0-) - \ldots f^{(n-1)}(0-) \; n \text{ derivations}$$

B.2 The Fourier Transform of a Continuous Time Signal

$$\Omega = 2\pi F$$

1.	$w(t) = \mathcal{F}^{-1}\{W(F)\} = \int_{-\infty}^{\infty} W(F) e^{j2\pi Ft} dF$	$\longleftrightarrow$	$W(F) = \mathcal{F}\{w(t)\} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi Ft} dt$
2.	$\sum_\nu a_\nu w_\nu(t)$	$\longleftrightarrow$	$\sum_\nu a_\nu W_\nu(F)$
3.	$w^*(-t)$	$\longleftrightarrow$	$W^*(F)$
4.	$W(t)$	$\longleftrightarrow$	$w(-F)$
5.	$w(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a\rvert} W\left(\frac{F}{a}\right)$
6.	$w(t - t_0)$	$\longleftrightarrow$	$W(F) \cdot e^{-j2\pi F t_0}$
7.	$w(t) \cdot e^{j2\pi F_0 t}$	$\longleftrightarrow$	$W(F - F_0)$
8.	$w^*(t)$	$\longleftrightarrow$	$W^*(-F)$
9.	$\frac{d^n w(t)}{dt^n}$	$\longleftrightarrow$	$(j2\pi F)^n \; W(F)$
10.	$\int_{-\infty}^{t} w(\tau) d\tau$	$\longleftrightarrow$	$\frac{1}{j2\pi F} \; W(F)$ if $W(F) = 0$ for $F = 0$
11.	$-j2\pi t \; w(t)$	$\longleftrightarrow$	$\frac{dw}{dF}$
12.	$w_1(t) * w_2(t)$	$\longleftrightarrow$	$W_1(F) \cdot W_2(F)$
13.	$w_1(t) \cdot w_2(t)$	$\longleftrightarrow$	$W_1(F) * W_2(F)$
14.	$\int_{-\infty}^{\infty} \lvert w(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert W(F)\rvert^2 dF$		Parseval's relation
15.	$\int_{-\infty}^{\infty} w_1(t) \cdot w_2(t) dt = \int_{-\infty}^{\infty} W_1(F) \cdot W_2^*(F) dF$		$w_1(t), \; w_2(t)$ real
16.	$\delta(t)$	$\longleftrightarrow$	1
17.	1	$\longleftrightarrow$	$\delta(F)$
18.	$u(t)$	$\longleftrightarrow$	$\frac{1}{j2\pi F} + \frac{1}{2} \; \delta(F)$
19.	$e^{-at} u(t)$	$\longleftrightarrow$	$\frac{1}{a + j\Omega}$

20.	$e^{-a\|t\|}$	$\longleftrightarrow$	$\frac{2a}{a^2+\Omega^2}$
21.	$e^{j2\pi F_0 t}$	$\longleftrightarrow$	$\delta(F-F_0)$
22.	$\sin 2\pi F_0 t$	$\longleftrightarrow$	$j\ \frac{1}{2}\ \{\delta(F+F_0)-\delta(F-F_0)\}$
23.	$\sin 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\frac{\Omega_0}{\Omega_0^2-\Omega^2}+j\ \frac{1}{4}\ \{\delta(F+F_0)-\delta(F-F_0)\}$
24.	$\cos 2\pi F_0 t$	$\longleftrightarrow$	$\frac{1}{2}\ \{\delta(F+F_0)+\delta(F-F_0)\}$
25.	$\cos 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\frac{j\Omega}{\Omega_0^2-\Omega^2}+\frac{1}{4}\{\delta(F+F_0)+\delta(F-F_0)\}$
26.	$\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-t^2/2\sigma^2}$	$\longleftrightarrow$	$e^{-(\Omega\sigma)^2/2}$
27.	$e^{-at}\sin 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\dfrac{\Omega_0}{(j\Omega+a)^2+(\Omega_0)^2}$
28.	$e^{-a\|t\|}\sin 2\pi F_0\|t\|$	$\longleftrightarrow$	$\dfrac{2\Omega_0(\Omega_0^2+a^2-\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$
29.	$e^{-at}\cos 2\pi F_0 t \cdot u(t)$	$\longleftrightarrow$	$\dfrac{j\Omega+a}{(j\Omega+a)^2+(\Omega_0)^2}$
30.	$e^{-a\|t\|}\cos 2\pi F_0 t$	$\longleftrightarrow$	$\dfrac{2a(\Omega_0^2+a^2+\Omega^2)}{(\Omega^2+a^2-\Omega_0^2)^2+4a^2\Omega_0^2}$
31.	$\mathrm{rect}(at)=\begin{cases}1 \text{ for } \|t\|<\frac{1}{2a}\\ 0 \text{ Else}\end{cases}$	$\longleftrightarrow$	$\frac{1}{a}\ \mathrm{sinc}(\frac{F}{a})\quad a>0$
32.	$\mathrm{sinc}(at)=\frac{\sin(\pi at)}{\pi at}$	$\longleftrightarrow$	$\frac{1}{a}\ \mathrm{rect}(\frac{F}{a})\quad a>0$
33.	$\mathrm{rep}_T(w(t))=\sum_{m=-\infty}^{\infty}w(t-mT)$	$\longleftrightarrow$	$\frac{1}{\|T\|}\ \mathrm{comb}_{1/T}(W(F))$
34.	$\|T\|\mathrm{comb}_T(w(t))=$ $\|T\|\sum_{m=-\infty}^{\infty}w(mT)\delta(t-mT)$	$\longleftrightarrow$	$\mathrm{rep}_{1/T}(W(F))$
35.	$\sum_{n=-\infty}^{\infty}c_n\delta(t-nT)$	$\longleftrightarrow$	$\sum_{n=-\infty}^{\infty}\frac{1}{T}\ c_n\delta(F-\frac{n}{T})=\sum c_n e^{-j2\pi nTF}$

B.3 The Z-transform

B.3.1 The Z-transform of causal signals

1.	$\mathcal{X}(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$	Transform
2.	$x(n) = Z^{-1}[\mathcal{X}(z)] = \frac{1}{2\pi j} \int_\Gamma \mathcal{X}(z)z^{n-1}dz$	Invers transform
3.	$\sum_\nu a_\nu x_\nu(n) \longleftrightarrow \sum_\nu a_\nu \mathcal{X}_\nu(z)$	Linearity
4.	$x(n-n_0) \longleftrightarrow z^{-n_0}\mathcal{X}(z)$	Shift (n_0 positive or negative integer)
5.	$nx(n) \longleftrightarrow -z\ \frac{d}{dz}\ \mathcal{X}(z)$	Multiplication with n
6.	$a^n x(n) \longleftrightarrow \mathcal{X}\left(\frac{z}{a}\right)$	Scaling
7.	$x(-n) \longleftrightarrow \mathcal{X}\left(\frac{1}{z}\right)$	Reflection of the time sequence
8.	$\left[\sum_{\ell=-\infty}^{n} x(\ell)\right] \longleftrightarrow \frac{z}{z-1}\ \mathcal{X}(z)$	Summation
9.	$x * y \longleftrightarrow \mathcal{X}(z) \cdot \mathcal{Y}(z)$	Convolution
10.	$x(n) \cdot y(n) \longleftrightarrow \frac{1}{2\pi j} \int_\Gamma \mathcal{Y}(\xi)\mathcal{X}\left(\frac{z}{\xi}\right) \xi^{-1}d\xi$	Product
11.	$x(0) = \lim_{z\to\infty} \mathcal{X}(z)$ (if the limit value exist)	Initial value theorem
12.	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\mathcal{X}(z)$ (if the unit cirle is included in the ROC)	End value theorem
13.	$\sum_{\ell=-\infty}^{\infty} x(\ell)y(\ell) = \frac{1}{2\pi j} \int_\Gamma x(z)y\left(\frac{1}{z}\right) z^{-1}dz$	Parseval's teorem for real time sequences
14.	$\sum_{\ell=-\infty}^{\infty} x^2(\ell) = \frac{1}{2\pi j} \int_\Gamma \mathcal{X}(z)\mathcal{X}(z^{-1})z^{-1}dz$	– " –

	Sequence		Transform
	$x(n)$	$\longleftrightarrow$	$\mathcal{X}(z)$
15.	$\delta(n)$	$\longleftrightarrow$	1
16.	$u(n)$	$\longleftrightarrow$	$\frac{1}{1-z^{-1}}$
17.	$nu(n)$	$\longleftrightarrow$	$\frac{z^{-1}}{(1-z^{-1})^2}$
18.	$\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{1-\alpha z^{-1}}$
19.	$(n+1)\alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^2}$
20.	$\frac{(n+1)(n+2)\ldots(n+r-1)}{(r-1)!}\ \alpha^n u(n)$	$\longleftrightarrow$	$\frac{1}{(1-\alpha z^{-1})^r}$
21.	$\alpha^n \cos\beta n\ u(n)$	$\longleftrightarrow$	$\frac{1-z^{-1}\alpha\cos\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
22.	$\alpha^n \sin\beta n\ u(n)$	$\longleftrightarrow$	$\frac{z^{-1}\alpha\sin\beta}{1-z^{-1}2\alpha\cos\beta+\alpha^2 z^{-2}}$
23.	$\mathbf{F}^n u(n)$	$\longleftrightarrow$	$(\mathbf{I}-z^{-1}\mathbf{F})^{-1}$

B.3.2 Single-Sided Z-transform of non-causal signals

Notation

$\mathcal{X}^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$	Single-sided z-transform, $x(n)$ not necessarily causal.
$\mathcal{X}(z) = \mathcal{X}^+(z)$	For causal signals

Shifting $x(n)$yields:
i) one step shift

$$
\begin{aligned}
x(n-1) &\longleftrightarrow z^{-1}\mathcal{X}^+(z) + x(-1) \\
x(n+1) &\longleftrightarrow z\mathcal{X}^+(z) - x(0)\cdot z
\end{aligned}
$$

ii) n_0 step shift ($n_0 \geq 0$)

$$
\begin{aligned}
x(n-n_0) &\longleftrightarrow z^{-n_0}\mathcal{X}^+(z) + x(-1)z^{-n_0+1} + \\
&\qquad +x(-2)z^{-n_0+2} + \ldots + x(-n_0) \\
x(n+n_0) &\longleftrightarrow z^{n_0}\mathcal{X}^+(z) - x(0)z^{n_0} - x(1)z^{n_0-1} - \ldots - x(n_0-1)z
\end{aligned}
$$

B.4 Fourier Transform for Discrete Time Signal

1.	$X(f) = \mathcal{F}(x(n)) =$ $= \sum_{\ell=-\infty}^{\infty} x(\ell) e^{-j2\pi f \ell} \quad \omega = 2\pi f$		Transform
2.	$x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df =$ $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(f) e^{j\omega n} d\omega$		Inverse transform
3.	$\sum a_\nu x_\nu(n)$	$\longleftrightarrow$	$\sum_\nu a_\nu X_\nu(f)$ Linearity
4.	$x(n - n_0)$	$\longleftrightarrow$	$X(f) \cdot e^{-j2\pi f n_0}$ Shift
5.	$x(n) e^{j2\pi f_0 n}$	$\longleftrightarrow$	$X(f - f_0)$ Frequency translation
6.	$x(n) \cdot \cos 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2} \left[X(f - f_0) + X(f + f_0) \right]$ Modulation
7.	$x(n) \cdot \sin 2\pi f_0 n$	$\longleftrightarrow$	$\frac{1}{2j} \left[X(f - f_0) - X(f + f_0) \right]$ Modulation
8.	$x * y$	$\longleftrightarrow$	$X(f) \cdot Y(f)$ Convolution
9.	$x \cdot y$	$\longleftrightarrow$	$\int_{-1/2}^{1/2} X(\lambda) \cdot Y(f - \lambda) d\lambda$ Product
10.	$\sum_{\ell=-\infty}^{\infty} x(\ell) y(\ell) =$ $= \int_{-1/2}^{1/2} X(f) Y^*(f) df$		Parseval's teorem for real time sequences
11.	$X(f) = \mathcal{X}(e^{j\omega})$		if $x(n) = 0$ for $n < n_0$ and $\sum_{\ell=-\infty}^{\infty} \lvert x(\ell) \rvert^2 < \infty$ (Valid for for example: 18,19,20,21 och 22 in the Z-transform table for $\lvert\alpha\rvert < 1$)
12.	$\delta(n)$	$\longleftrightarrow$	1
13.	$\delta(n - n_0)$	$\longleftrightarrow$	$e^{-j\omega n_0}$
14.	$1 \ \forall n$	$\longleftrightarrow$	$\sum_{p=-\infty}^{\infty} \delta(f - p)$
15.	$u(n)$	$\longleftrightarrow$	$\frac{1}{2} \sum_{p=-\infty}^{\infty} \delta(f - p) + \frac{1}{2} + \frac{1}{j \cdot 2 \cdot \tan(\pi f)}$

16. $2f_1 \cdot \mathrm{sinc}(2f_1 \cdot n) = 2f_1 \; \dfrac{\sin(2\pi f_1 n)}{2\pi f_1 n}$

$$\longleftrightarrow \mathrm{rect}_p\left(\frac{f}{2f_1}\right) = \begin{cases} 1 & |f-n| < f_1 < 1/2 \;,\; n \text{ integer} \\ 0 & \text{Else} \end{cases}$$

Ideal LP filter

17. $4f_1 \mathrm{sinc}(2f_1 n)\cos(2\pi f_0 n)$

$$\longleftrightarrow \mathrm{rect}_p\left(\frac{f-f_0}{2f_1}\right) + \mathrm{rect}_p\left(\frac{f+f_0}{2f_1}\right) \text{ Ideal BP Filter}$$

18. $\dfrac{2\pi f_1 n \cos 2\pi f_1 n - \sin 2\pi f_1 n}{\pi n^2}$

$$\longleftrightarrow (j2\pi f)_p = \begin{cases} j2\pi(f-n) & |f-n| < f_1 < 1/2 \;,\; n \text{ integer} \\ 0 & \text{Else} \end{cases}$$

"Derivating" system

19. $\cos(2\pi f_0 n) \longleftrightarrow \dfrac{1}{2} \sum_{p=-\infty}^{\infty} [\delta(f - f_0 - p) + \delta(f + f_0 - p)]$

20. $\alpha^{|n|} \longleftrightarrow \dfrac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos 2\pi f}$

21. $\alpha^{|n|}\cos(2\pi f_0 n)$

$$\longleftrightarrow \frac{1-\alpha^2}{2}\left[\frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f+f_0)} + \frac{1}{1+\alpha^2-2\alpha\cos 2\pi(f-f_0)}\right]$$

22. $p_r(n) = \begin{cases} 1 & |n| \le \dfrac{M-1}{2} \\ 0 & \text{Else} \end{cases} \quad M \text{ odd}$

$$\longleftrightarrow P_r(f) = \frac{\sin(\pi f M)}{\sin(\pi f)} \text{ Rectangular window}$$

B.5 Some DFT Properties

Time	Frequency
$x(n),\ y(n)$	$X(k),\ Y(k)$
$x(n) = x(n+N)$	$X(k) = X(k+N)$
$x(N-1)$	$X(N-k)$
$x((n-1))_{<N>}$	$X(k)e^{-j2\pi k1/N}$
$x(n)e^{j2\pi 1n/N}$	$X((k-1))_{<N>}$
$x^*(n)$	$X^*(N-k)$
$x_1(n) \circledast x_2(n)$	$X_1(k)X_2(k)$
$x(n) \circledast y^*(-n)$	$X(k)Y^*(k)$
$x_1(n)x_2(n)$	$\frac{1}{N}\ X_1(k)\ N\ X_2(k)$
$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$