Solutions Multivariable calculus, 2008-10-31.

1. The function $f(x,y) = e^{x^2+2x+y^2}$ is defined on the disk $D = \{(x,y)\,\epsilon\,\mathbb{R}^2 |\, x^2+2x+y^2 \leq 0\}$.

   (a) Calculate the maximum directional derivative of $f(x,y)$ at the point $(-\frac{1}{2},\frac{1}{2})$. *(2p)*

   Answer:
   The maximum value of the directional derivative at the given point is:
   $|\nabla f(-\frac{1}{2},\frac{1}{2})| = |(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})_{(-\frac{1}{2},\frac{1}{2})}| = |((2x+2)\,e^{x^2+2x+y^2}, 2y\,e^{x^2+2x+y^2})_{(-\frac{1}{2},\frac{1}{2})}| =$
   $|(e^{-\frac{1}{2}}, e^{-\frac{1}{2}})| = \sqrt{\frac{2}{e}}$.

   (b) Calculate the absolute minimum and maximum values of $f(x,y)$ (on D). *(3p)*

   Answer:
   We first establish if there are any interior critical points:

   $$\begin{array}{rcl} f_x & = & 0 \\ f_y & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} (2x+2)e^{x^2+2x+y^2} & = & 0 \\ 2ye^{x^2+2x+y^2} & = & 0 \end{array} \Leftrightarrow (x,y) = (-1,0).$$ This is an interior point of D.

   We then observe that, on the boundary of D (that is, when $x^2+2x+y^2 = 0$), f has the constant value $f(x,y) = e^0 = 1$.
   Since $f(-1,0) = e^{-1} < 1$, we can infer that $f_{min} = e^{-1}$ and $f_{max} = 1$.
   If we rewrite the function as $f(x,y) = e^{(x+1)^2+y^2-1}$ the result above can be arrived at even more directly.

   (c) Calculate $\iint\limits_D f(x,y)\,dx\,dy$. *(3p)*

   Answer:
   Taking our cue from the last comment above we set: $x+1 = r\cos\theta$, $y = r\sin\theta$. This gives:
   $x^2+2x+y^2 \leq 0 \Leftrightarrow (x+1)^2+y^2 \leq 1 \Leftrightarrow 0 \leq r \leq 1$. This gives

   $$\iint\limits_D f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2-1}\,r\,dr\,d\theta = 2\pi\,e^{-1}\left[\frac{1}{2}e^{r^2}\right]_0^1 = \pi(1-e^{-1}).$$

   Notice that $A_D\,f_{min} < \pi(1-e^{-1}) < A_D\,f_{max}$, where $A_D = \pi$ is the area of the disk.

2. Find and classify all critical points of the function $g(x,y) = x^2y + y^2 + 2xy$. *(3p)*

   Answer:
   We first find all critical points:

   $$\begin{array}{rcl} g_x & = & 0 \\ g_y & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2y(x+1) & = & 0 \\ x^2+2y+2x & = & 0 \end{array} \Leftrightarrow (y=0 \vee x=-1) \wedge x^2+2y+2x = 0 \Leftrightarrow$$
   $(x,y) = (0,0) \vee (x,y) = (-2,0) \vee (x,y) = (-1,\frac{1}{2})$.

   Also, we need the second derivatives: $g_{xx} = 2y$, $g_{yy} = 2$, $g_{xy} = 2x+2$.
   An expansion to second order around the critical points then yields:

   $(x,y) = (0,0):\quad g(0+h,0+k) \simeq g(0,0) + \frac{1}{2}\Big(g_{xx}(0,0)h^2 + 2g_{xy}(0,0)hk + g_{yy}(0,0)k^2\Big) =$

   $2hk + k^2 = (h+k)^2 - h^2$. This means that $(0,0)$ is a saddle-point.
   (This result can also be obtained directly from the functional form of g).

   $(x,y) = (-2,0):\quad g(-2+h,0+k) \simeq g(-2,0) + \frac{1}{2}\Big(g_{xx}(-2,0)h^2 + 2g_{xy}(-2,0)hk + g_{yy}(-2,0)k^2\Big) =$
   $-2hk + k^2 = (h-k)^2 - h^2$. We see that $(-2,0)$ is also a saddle-point.

$(x,y)=(-1,\frac{1}{2}):\quad g(-1+h,\frac{1}{2}+k)\simeq g(-1,\frac{1}{2})+\frac{1}{2}\Big(g_{xx}(-1,\frac{1}{2})h^2+2g_{xy}(-1,\frac{1}{2})hk+g_{yy}(-1,\frac{1}{2})k^2\Big)=$ $-\frac{1}{4}+\frac{1}{2}h^2+k^2$. From this we conclude that $(-1,\frac{1}{2})$ is a local minimum point.

3. Calculate $\displaystyle\iint_A \sqrt{\frac{x}{y}}\,dx\,dy$, where A is bounded by the curves $x^2y=1$, $y=1$ and $x=4$. *(3p)*

Answer:
A sketch of the curves will show that the region A can be described as $A:\ \dfrac{1}{x^2}\le y\le 1,\ 1\le x\le 4$.

This gives $\displaystyle\iint_A \sqrt{\frac{x}{y}}\,dx\,dy = \int_1^4 \sqrt{x}\Big(\int_{\frac{1}{x^2}}^1 \frac{1}{\sqrt{y}}\,dy\Big)\,dx = \int_1^4 \sqrt{x}\,\Big[2\sqrt{y}\Big]_{\frac{1}{x^2}}^1\,dx =$

$$2\int_1^4 \Big(\sqrt{x}-\frac{1}{\sqrt{x}}\Big)\,dx = 2\left[\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}\right]_1^4 = \frac{16}{3}.$$

4. Calculate $\displaystyle\iint_D (x^2-y^2)\,\ln(x+y)\,dx\,dy$, $D=\{(x,y)\,\epsilon\,\mathbb{R}^2|\,0\le x-y\le 1,\ \ 1\le x+y\le 2\}$. *(3p)*

Answer:
The 'tilted' square D in the $xy-$plane can be transformed into the square $0\le u\le 1,\ 1\le v\le 2$ in the uv-plane by the substitution:

$u=x-y,\ v=x+y,$ which gives $\displaystyle\frac{\partial(x,y)}{\partial(u,v)}=\Big(\frac{\partial(u,v)}{\partial(x,y)}\Big)^{-1}=\frac{1}{2}.$

Inserted into the integral:

$$\iint_D (x^2-y^2)\,\ln(x+y)\,dx\,dy = \frac{1}{2}\int_0^1\int_1^2 u\,v\,\ln v\,dv\,du =$$

$$\frac{1}{2}\int_0^1 u\,du\int_1^2 v\,\ln v\,dv = \frac{1}{2}\Big[\frac{1}{2}u^2\Big]_0^1\Big(\Big[\frac{1}{2}v^2\ln v\Big]_1^2 - \int_1^2 \frac{1}{2}v^2\,\frac{1}{v}\,dv\Big) =$$

$$\frac{1}{4}\Big(2\,\ln 2 - \Big[\frac{1}{4}v^2\Big]_1^2\Big) = \frac{\ln 2}{2}-\frac{3}{16}.$$

5. Calculate $\displaystyle\iiint_B \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$, where B is the solid sphere $x^2+y^2+z^2\le 3$. *(3p)*

Answer:
Using spherical coordinates:

$$\iiint_B \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{3}} \frac{1}{\rho}\,\rho^2\,d\rho\,\sin\phi\,d\phi\,d\theta =$$

$$4\pi\int_0^{\sqrt{3}} \rho\,d\rho = 4\pi\left[\frac{1}{2}\,\rho^2\right]_0^{\sqrt{3}} = 6\pi.$$

6. Find the absolute minimum and maximum values of $f(x,y)=x^2+y^2$
subject to the constraint $5x^2-6xy+5y^2=4$. *(5p)*

Answer:
Alt. 1 Using the method of Lagrange multipliers (let $g(x,y,z)=5x^2-6xy+5y^2-4=0$ represent the constraint) :

$$\begin{array}{rcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \end{array} \Leftrightarrow \begin{array}{rcl} 2x &=& \lambda(10x-6y) \\ 2y &=& \lambda(10y-6x) \end{array} \Rightarrow$$

(multiply the two equations with y and x, respectively)

$$\begin{array}{rcl} 2xy &=& \lambda(10x-6y)y \\ 2yx &=& \lambda(10y-6x)x \end{array} \Rightarrow \lambda(10xy-6y^2)-\lambda(10yx-6x^2) = 0 \Leftrightarrow \lambda=0\ \vee\ y=x\ \vee y=-x.$$

Now, $\lambda = 0$ gives $(x, y) = (0, 0)$ which does not satisfy the constraint. This leaves us with the alternatives:

$y = x$ into the constraint: $5x^2 - 6x^2 + 5y^2 - 4 = 0 \Leftrightarrow x = \pm 1$.
Interesting points: $(-1, -1)$ and $(1, 1)$.

$y = -x$ into the constraint: $5x^2 - 6x(-x) + 5y^2 - 4 = 0 \Leftrightarrow x = \pm\frac{1}{2}$.
Interesting points: $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$.

Comparison: $f(-1,-1) = f(1,1) = 2,\ f(-\frac{1}{2}, \frac{1}{2}) = f(\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2}$.

Conclusion: $f_{min} = \frac{1}{2},\ f_{max} = 2$.

7. Calculate $\iiint_K z\sqrt{x^2+y^2}\, dx\, dy\, dz$, $K = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \le z \le 2 - x^2 - y^2\}$. *(5p)*

Answer:
Alternative 1:

With $r = \sqrt{x^2+y^2}$, we get $r \le z \le 2 - r^2 \Rightarrow r^2 + r - 2 \le 0 \Leftrightarrow 0 \le r \le 1$ (since $r \ge 0$).
This gives:

$$\iiint_K z\sqrt{x^2+y^2}\, dx\, dy\, dz = \int_0^{2\pi}\int_0^1 r\Big(\int_r^{2-r^2} z\, dz\Big) r dr\, d\theta = 2\pi\int_0^1 r^2\Big[\frac{1}{2}z^2\Big]_r^{2-r^2} dr =$$

$$\pi\int_0^1 r^2((2-r^2)^2 - r^2)\, dr = \pi\int_0^1 (r^6 - 5r^4 + 4r^2)\, dr = \pi\Big[\frac{1}{7}r^7 - r^5 + \frac{4}{3}r^3\Big]_0^1 = \frac{10\pi}{21}.$$

Alternative 2:

A 'slize' through K parallell to the xy-plane will be a circular disk with radius $r = z$ for $0 \le z \le 1$ and $r = \sqrt{2-z}$ for $1 \le z \le 2$. This gives:

$$\iiint_K z\sqrt{x^2+y^2}\, dx\, dy\, dz = 2\pi\int_0^1 z\Big(\int_0^z r\, r\, dr\Big) dz + 2\pi\int_1^2 z\Big(\int_0^{\sqrt{2-z}} r\, r\, dr\Big) dz =$$

$$\frac{2\pi}{3}\int_0^1 z^3\, dz + \frac{2\pi}{3}\int_1^2 z\,(2-z)^{\frac{3}{2}}\, dz = \frac{10\pi}{21}.$$