Solutions Multivariable calculus, 2007-03-15.

1. Find an equation of the plane tangent to the surface $\arctan(x+y-z^2)+x+y=4$ at the point $(1,3,2)$. *(2p)*

   Answer:
   Set: $F(x,y,z) = \arctan(x+y-z^2)+x+y$.

   Then: $\nabla F(1,3,2) = (F_x, F_y, F_z)_{(1,3,2)} =$

   $$\Big(\frac{1}{1+(x+y-z^2)^2}+1, \frac{1}{1+(x+y-z^2)^2}+1, \frac{-2z}{1+(x+y-z^2)^2}\Big)_{(1,3,2)} = (2,2,-4) = 2(1,1,-2).$$

   We choose $\boldsymbol{n} = (1,1,-2)$ as the normal vector for the tangent plane.

   The equation of the plane tangent is given by :

   $$\boldsymbol{n}\cdot\Big((x,y,z)-(1,3,2)\Big) = 0 \Leftrightarrow (x-1)+(y-3)+(-2)(z-2) = 0 \Leftrightarrow x+y-2z = 0.$$

2. Calculate the directional derivative of $f(x,y,z) = \dfrac{x-y-z}{x^2+y^2}$ at the point $P=(1,1,-1)$ and in the direction (from P) towards $Q=(2,3,1)$. *(2p)*

   Answer:
   We first find a normalized direction vector (pointing from P to Q) :

   $\boldsymbol{u} = \dfrac{(2,3,1)-(1,1,-1)}{|(2,3,1)-(1,1,-1|} = \dfrac{1}{3}(1,2,2)$. The directional derivative is then :

   $$D\, f_{\boldsymbol{u}}(1,1,-1) = \boldsymbol{u}\cdot\nabla f(1,1,-1) = \boldsymbol{u}\cdot(f_x,f_y,f_z)_{(1,1,-1)} =$$

   $$\boldsymbol{u}\cdot\Big(\frac{(x^2+y^2)-2x(x-y-z)}{(x^2+y^2)^2}, \frac{-(x^2+y^2)-2y(x-y-z)}{(x^2+y^2)^2}, \frac{-1}{x^2+y^2}\Big)_{(1,1,-1)} =$$

   $$\frac{1}{3}(1,2,2)\cdot(0,-1,-\frac{1}{2}) = -1.$$

3. The equation $x^3+yz-xz^2 = 2$ implicitly defines a function $z=f(x,y)$ for which $f(-2,1)=2$. Calculate $f_x(-2,1)$. *(2p)*

   Answer:
   Taking the partial derivative of both sides of the equation with respect to x:

   $$\frac{\partial}{\partial x}\Big(x^3+yz-xz^2\Big) = 0 \Leftrightarrow 3x^2+yz_x-z^2-2xzz_x = 0 \Leftrightarrow (y-2xz)z_x = z^2-3x^2- \Leftrightarrow z_x = \frac{z^2-3x^2}{y-2xz}.$$

   Inserting $x=-2,\ y=1,\ z=2$ we obtain: $f_x(-2,1) = z_x(-2,1) = -\dfrac{8}{9}$.

4. Calculate $\displaystyle\iint_A x\,dx\,dy$, $A = \{(x,y)\,\epsilon\,\mathbb{R}^2 |\, x^2+y^2 \le 3y\ ,\ \ x\ge 0\}$. *(3p)*

   Answer:
   Using polar coordinates: $x^2+y^2\le 3y \Leftrightarrow r^2 \le 3r\sin\theta \Rightarrow r\le 3\sin\theta$.

   Also, since $x,\ y\ge 0$, we have: $0\le\theta\le\dfrac{\pi}{2}$. This gives:

   $$\iint_A x\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{3\sin\theta} r\cos\theta\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\Big[\frac{1}{3}r^3\Big]_0^{3\sin\theta}\cos\theta\,d\theta = 9\int_0^{\frac{\pi}{2}}\sin^3\theta\,\cos\theta\,d\theta =$$

$\Big[u = \sin\theta\Big] \;=\; 9 \int\limits_0^1 u^3\,du \;=\; \frac{9}{4}.$

5. Calculate $\displaystyle\iint\limits_D \ln(x^2 - 4y^2)\,dx\,dy$, $D \;=\; \{(x, y)\,\epsilon\,\mathbb{R}^2 |\; 1 \le x + 2y \le 2,\;\; 1 \le x - 2y \le 2\}$. *(3p)*

   Answer:
   The parallellogram D in the $xy-$plane can be transformed into a square D_{uv} : $1 \le u \le 2,\; 1 \le v \le 2$ by the substitution:

   $u = x + 2y,\;\; v = x - 2y,$ which gives $\displaystyle\frac{\partial(x, y)}{\partial(u, v)} \;=\; \Big(\frac{\partial(u, v)}{\partial(x, y)}\Big)^{-1} \;=\; -\frac{1}{4}.$

   The integrand then becomes: $\ln(x^2 - 4y^2) \;=\; \ln\Big((x + 2y)(x - 2y)\Big) \;=\; \ln(u\,v) \;=\; \ln u + \ln v.$
   Inserted into the integral:

   $$\iint\limits_D \ln(x^2 - 4y^2)\,dx\,dy \;=\; \int\limits_1^2 \int\limits_1^2 (\ln u + \ln v)\,| - \frac{1}{4}|\,du\,dv \;=\; \text{(Due to symmetry)}$$

   $$=\; 2 \cdot \frac{1}{4} \int\limits_1^2 \ln u\,du \int\limits_1^2 dv \;=\; \frac{1}{2} \int\limits_1^2 \ln u\,du \;=\; \frac{1}{2}\Big[u \ln u - u\Big]_1^2 \;=\; \ln 2 - \frac{1}{2}.$$

6. Calculate the area of the part of the sphere $x^2 + y^2 + z^2 \;=\; 1$ for which $z \ge \frac{2}{3}$. *(3p)*

   Answer:
   Alt. 1: On the surface to be calculated $\frac{2}{3} \le z \le 1$, which means that the projection S_{xy} of the surface is given by $x^2 + y^2 \le \frac{5}{9} \;\Rightarrow\; r \le \frac{\sqrt{5}}{3}$. Also, on the surface $z = \sqrt{1 - x^2 - y^2}$. This gives :

   $$\iint\limits_S dS \;=\; \iint\limits_{S_{xy}} \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy \;=\; \int\limits_0^{2\pi} \int\limits_0^{\frac{\sqrt{5}}{3}} \frac{1}{\sqrt{1 - r^2}}\,r\,dr\,d\theta \;=\; 2\pi\,\Big[-\sqrt{1 - r^2}\Big]_0^{\frac{\sqrt{5}}{3}} \;=\; \frac{2\pi}{3}.$$

   Alt. 2: In spherical coordinates we have: $dS \;=\; \sin\phi\,d\phi\,d\theta$. Also, the range in z (see above) implies $\frac{2}{3} \le \cos\phi \le 1$:

   $$\iint\limits_S dS \;=\; \int\limits_0^{2\pi} \int\limits_0^{\arccos(\frac{2}{3})} \sin\phi\,d\phi\,d\theta \;=\; \Big[u = \cos\phi\Big] \;=\; 2\pi \int\limits_1^{\frac{2}{3}} (-du) \;=\; 2\pi\Big[u\Big]_{\frac{2}{3}}^1 \;=\; \frac{2\pi}{3}.$$

7. Find the (absolute) minimum and maximum values of $g(x, y) \;=\; x^2 - y^2 - 2x + 4y$ on the set $\Delta \;=\; \{(x, y)\,\epsilon\,\mathbb{R}^2 \;|\; 0 \le y \le x + 2,\;\; x \le 2\}$. *(5p)*

   Answer:
   First finding critical points:

   $$\begin{array}{lcl} f_x &=& 0 \\ f_y &=& 0 \end{array} \;\Leftrightarrow\; \begin{array}{rcl} 2x - 2 &=& 0 \\ -2y + 4 &=& 0 \end{array} \;\Leftrightarrow (x, y) = (1, 2). \quad \text{This is an interior point of } \Delta.$$

   Next we consider the values of g on the three straight-line segments that constitute the boundary of the triangular region Δ :

   1: $-2 \le x \le 2,\;\; y = 0$: $f(x, 0) = x^2 - 2x = g(x) \;\Rightarrow\; g'(x) = 2x - 2 = 0 \;\Rightarrow\; x = 1.$
   Interesting points: $(-2, 0),\; (1, 0),\; (2, 0).$

   2: $0 \le y \le 4,\; x = 2$: $f(2, y) = -y^2 + 4y = h(y) \;\Rightarrow\; h'(y) = -2y + 4 = 0 \;\Rightarrow\; y = 2.$
   Additional interesting points: $(2, 2), (2, 4).$

3: $y = x+2,\ -2 \le x \le 2$: $f(x, x+2) = -2x+4$.
The function is thus monotonously decreasing along this line segment. Furthermore, the corners are already accounted for. So, this segment does not yield any additional interesting points.

Finally, comparing the values of f at the single interior critical point and the boundary points given above: $f(1,2) = 3,\ f(-2,0) = 8,\ f(2,0) = 0,\ f(1,0) = -1,\ f(2,2) = 4, f(2,4) = 0$.

Conclusion: $f_{max} = 8,\ f_{min} = -1$.

8. Find the (absolute) minimum and maximum values of $f(x,y,z) = x - y + z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$. *(5p)*

Answer:
Alt. 1: Using the method of Lagrange multipliers (let $g(x,y,z) = x^2+y^2+z^2-1 = 0$ represent the constraint):

$$\begin{array}{rcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \\ f_z &=& \lambda g_z \end{array} \Leftrightarrow \begin{array}{rcl} 1 &=& 2\lambda x \\ -1 &=& 2\lambda y \\ 2z &=& 2\lambda z \end{array} \Rightarrow (x,y,z) = (\frac{1}{2\lambda}, -\frac{1}{2\lambda}, 0), \quad \text{provided that } \lambda \neq 0, 1.$$

Inserted into the constraint: $g(\frac{1}{2\lambda}, -\frac{1}{2\lambda}, 0) = 1 \Leftrightarrow \lambda = \pm\frac{1}{\sqrt{2}}$. This gives the following points: $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0),\ (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.

$\lambda = 0$ obviously does not give a solution to the equations above; whereas $\lambda = 1$ gives $(x,y) = (\frac{1}{2}, -\frac{1}{2})$. Upon insertion into the constraint (again !) we get $z = \pm\frac{1}{\sqrt{2}}$. That is, we obtain the two additional points:

$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}),\ (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}})$.

A comparison finally gives $f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) = \sqrt{2},\ f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = -\sqrt{2},\ f(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}) = \frac{3}{2}$.

Conclusion: $f_{max} = \frac{3}{2},\ f_{min} = -\sqrt{2}$.

Alt. 2: Due to the constraint $z^2 = 1 - x^2 - y^2$ we can write

$$f(x, y, \pm\sqrt{1-x^2-y^2}) = -x^2 - y^2 + x - y + 1 = h(x,y).$$

The function h is defined on the bounded set $\Gamma:\ x^2+y^2 \le 1$. We will then find a (single) interior critical point at $(\frac{1}{2}, -\frac{1}{2})$. The boundary of Γ can be parametrized as $(x,y) = (\cos t, \sin t)$. This means that $h(x,y) = \cos t - \sin t = j(t)$. The function j has a range $-\sqrt{2} \le j(t) \le \sqrt{2}$. The values above are therefore reproduced.

9. Calculate $\displaystyle\iiint\limits_K \frac{z}{1+x^2+y^2}\,dx\,dy\,dz$, $K = \{(x,y,z)\,\epsilon\,\mathbb{R}^3\,|\,x^2+y^2+z^2 \le 1,\; z \ge \sqrt{x^2+y^2}\}$. *(5p)*

Answer:
The limits in z, $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2} \;\Leftrightarrow\; r \le z \le \sqrt{1-r^2}$ also imply that $r \le \sqrt{1-r^2} \Rightarrow 0 \le r \le \dfrac{1}{\sqrt{2}}$.

We can then perform the triple integral by first integrating in the z-direction followed by an integral over the projection of K on the xy-plane:

$$\iiint\limits_K \frac{z}{x^2+y^2+1}\,dx\,dy\,dz \;=\; \iint\limits_{K_{xy}} \frac{1}{x^2+y^2+1}\Big(\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dz\Big)\,dx\,dy \;=$$

$$\iint\limits_{K_{xy}} \frac{1}{x^2+y^2+1}\left[\frac{1}{2}\,z^2\right]_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dx\,dy \;=\; \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{\frac{1}{\sqrt{2}}}\Big(\frac{-2r^2+1}{r^2+1}\Big)\,r\,dr\,d\theta \;=$$

$$\pi\int\limits_0^{\frac{1}{\sqrt{2}}}\Big(-2r + \frac{3r}{r^2+1}\Big)\,dr \;=\; \pi\Big[-r^2 + \frac{3}{2}\ln(r^2+1)\Big]_0^{\frac{1}{\sqrt{2}}} \;=\; \frac{\pi}{2}\Big(3\,\ln\big(\frac{3}{2}\big) - 1\Big).$$