Solutions Multivariable calculus, 2007-10-26.

1. Find an equation of the plane tangent to the surface $e^{2x+yz} - z = 0$ at the point $(0,0,1)$. *(2p)*

   Answer:
   Set: $F(x,y,z) = e^{2x+yz} - z.$

   Then: $\nabla F(0,0,1) = (F_x, F_y, F_z)_{(0,0,1)} = (2e^{2x+yz}, ze^{2x+yz}, ye^{2x+yz} - 1)_{(0,0,1)} = (2,1,-1).$

   The equation of the plane tangent is given by the equation provided :

   $$F_x(0,0,1)(x-0) + F_x(0,0,1)(y-0) + F_z(0,0,1)(z-1) = 0 \Leftrightarrow 2x + y - z + 1 = 0.$$

2. Find the following limit or show that it does not exist $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2+x^3-y^3}$. *(2p)*

   Answer:
   Using polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$. Note that $(x,y) \to 0 \Leftrightarrow r \to 0$:

   $$\frac{\sin(x^2+y^2)}{3x^2+3y^2+x^3-y^3} = \frac{\sin(r^2)}{3r^2 + r^3(\cos^3\theta - \sin^3\theta)} =$$
   $$\frac{1}{3} \cdot \frac{1}{1+\frac{1}{3}r(\cos^3\theta - \sin^3\theta)} \cdot \frac{\sin(r^2)}{r^2} \to \frac{1}{3}, \text{ as } r \to 0.$$

3. Find a point $P = (a,b)$ such that the directional derivatives of $f(x,y) = x^2 + yx$ at (a,b) are $f'_{\boldsymbol{u}}(a,b) = 8$ and $f'_{\boldsymbol{v}}(a,b) = 2$ in the directions $\boldsymbol{u} = (1,0)$ and $\boldsymbol{v} = (3,-4)$, respectively. *(2p)*

   Answer:
   Again using the given formula and normalized direction vectors:

   $$\begin{array}{rcl} f'_{\boldsymbol{u}}(a,b) &=& 8 \\ f'_{\boldsymbol{u}}(a,b) &=& 2 \end{array} \Leftrightarrow \begin{array}{rcl} (1,0)\cdot(2a+b,a) &=& 8 \\ (\frac{3}{5},-\frac{4}{5})\cdot(2a+b,a) &=& 2 \end{array} \Leftrightarrow \begin{array}{rcl} 2a+b &=& 8 \\ 2a+3b &=& 10 \end{array} \Leftrightarrow (a,b) = (\tfrac{7}{2}, 1).$$

4. The equation $\sqrt{x} + y^2 + z + \sin(xz) = 2$ implicitly defines a function $z = f(x,y)$ for which $f(1,1) = 0$. Calculate $f_x(1,1)$ and $f_y(1,1)$. *(3p)*

   Answer:
   $$\sqrt{x} + y^2 + z + \sin(xz) = 2 \Rightarrow \frac{\partial}{\partial x}\Big(\sqrt{x} + y^2 + z + \sin(xz)\Big) = 0 \Rightarrow$$
   $$\frac{1}{2\sqrt{x}} + z_x + (z + xz_x)\cos(xz) = 0 \Rightarrow z_x = -\frac{2z\sqrt{x}\cos(xz)+1}{2\sqrt{x}(x\cos(xz)+1)} \Rightarrow$$
   $$f_x(1,1) = \Big(-\frac{2z\sqrt{x}\cos(xz)+1}{2\sqrt{x}(x\cos(xz)+1)}\Big)_{(1,1,0)} = -\frac{1}{4}.$$
   $$\sqrt{x} + y^2 + z + \sin(xz) = 2 \Rightarrow \frac{\partial}{\partial y}\Big(\sqrt{x} + y^2 + z + \sin(xz)\Big) = 0 \Rightarrow$$
   $$2y + z_y + xz_y\cos(xz) = 0 \Rightarrow z_y = -\frac{2y}{x\cos(xz)+1} \Rightarrow$$
   $$f_y(1,1) = \Big(-\frac{2y}{x\cos(xz)+1}\Big)_{(1,1,0)} = -1.$$

5. Calculate $\iint_D \frac{2x+y}{(x-2y)^2+1}\,dx\,dy$, $D = \{(x,y)\,\epsilon\,\mathbb{R}^2 \mid 1 \le 2x+y \le 2,\ \ 0 \le x-2y \le 1\}$. *(3p)*

   Answer:
   The parallellogram D in the $xy-$plane can be transformed into a square $D_{uv} : 1 \le u \le 2,\ 0 \le v \le 1$ by the substitution:

$u = 2x + y,\ v = x - 2y,$ which gives $\dfrac{\partial(x,y)}{\partial(u,v)} = \left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\dfrac{1}{5}.$

Inserted into the integral:

$$\iint_D \frac{2x+y}{(x-2y)^2+1}\,dx\,dy = \int_0^1 \int_1^2 \frac{u}{v^2+1}\,|-\frac{1}{5}|\,du\,dv =$$

$$\frac{1}{5}\int_1^2 u\,du \int_0^1 \frac{1}{v^2+1}\,dv = \frac{1}{5}\Big[\frac{1}{2}u^2\Big]_1^2 \Big[\arctan v\Big]_0^1 = \frac{3\pi}{40}.$$

6. Calculate $\displaystyle\iint_S x\,dS,$ where the surface S is given by

$S:\ \boldsymbol{r} = (x,y,z) = (x,y,\sqrt{2x^2+2y^2}),\ \ x \le 1,\ -x \le y \le x.$ *(3p)*

Answer:
The projection of S on to the xy-plane is the triangle $S_{xy}:\ 0 \le x \le 1,\ -x \le y \le x.$
Set $h(x,y) = \sqrt{2x^2+2y^2}$ and we obtain (see 'Surface Integrals - Function graph') :

$$\iint_S x\,dS = \iint_{S_{xy}} x\,\sqrt{1+h_x^2+h_y^2}\,dxdy = \iint_{S_{xy}} x\,\sqrt{3}\,dx\,dy = \sqrt{3}\int_0^1 x\Big(\int_{-x}^{x} dy\Big)\,dx =$$

$$\sqrt{3}\int_0^1 2x^2\,dx = 2\sqrt{3}\,\Big[\frac{1}{3}x^3\Big]_0^1 = \frac{2}{\sqrt{3}}.$$

7. Find the (absolute) minimum and maximum values of $f(x,y) = 2x^2 + y^2 - 2xy - 2y$

on the set $\Delta = \{(x,y)\,\epsilon\,\mathbb{R}^2 \mid x^2 \le y \le 4,\ x \ge 0\}.$ *(5p)*

Answer:
First finding critical points:

$$\begin{array}{rcl} f_x &=& 0 \\ f_y &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 4x-2y &=& 0 \\ 2y-2x-2 &=& 0 \end{array} \Leftrightarrow (x,y) = (1,2).$$ This is an interior point of Δ.

Next we we consider the values of f on the boundary, which consists of three segments:

1. $0 \le x \le 2,\ y = x^2$: $f(x,x^2) = x^4 - 2x^3 = g_1(x) \Rightarrow g_1'(x) = 4x^3 - 6x^2 = 2x^2(2x-3).$

Then: $g_1'(x) = 0 \Leftrightarrow x = 0 \lor x = \frac{3}{2}$. Interesting points: $(0,0),\ (\frac{3}{2},\frac{9}{4}),\ (2,4).$

2. $y = 4,\ 0 \le x \le 2$: $f(x,4) = 2x^2 - 8x + 8 = g_2(x) \Rightarrow g_2'(x) = 4x - 8.$

$g_2'(x) = 0 \Leftrightarrow x = 2$. Additional interesting point: $(0,4)$.

3. $x = 0,\ 0 \le y \le 4$: $f(0,y) = y^2 - 2y = g_3(y) \Rightarrow g_3'(y) = 2y - 2.$

$g_3'(y) = 0 \Leftrightarrow y = 1$. Additional interesting point: $(0,1)$.

Comparison between the function values at these points:

$f(1,2) = -2,\ f(0,0) = 0,\ f(\frac{3}{2},\frac{9}{4}) = -\frac{27}{16},\ f(2,4) = 0,\ f(0,4) = 8,\ f(0,1) = -1.$

Conclusion: $f_{max} = 8,\ f_{min} = -2.$

8. Find the (absolute) minimum and maximum values of $f(x,y,z) = z$
subject to the constraints $x^2 + 2y^2 = 6$ and $x + y + z = 1.$ *(5p)*

Answer:
Alt. 1: Using the method of Lagrange multipliers (let $g(x,y,z) = x^2+2y^2-6 = 0$ and $h(x,y,z) = x+y+z-1 = 0$ represent the constraints) :

$$\begin{array}{rcl} f_x &=& \lambda g_x + \mu h_x \\ f_y &=& \lambda g_y + \mu h_y \\ f_z &=& \lambda g_z + \mu h_z \end{array} \Leftrightarrow \begin{array}{rcl} 0 &=& 2\lambda x + \mu \\ 0 &=& 4\lambda y + \mu \\ 1 &=& \mu \end{array} \Rightarrow 2\lambda x + \mu = 4\lambda y + \mu \Leftrightarrow \lambda = 0 \vee x = 2y.$$

$\lambda = 0$ does not give a solution of the equations above; while $x = 2y$ can be inserted into $g(x,y) = 0$:

$$g(2y,y,z) = 0 \Leftrightarrow y = \pm 1 \Leftrightarrow x = \pm 2 \Rightarrow \text{ (using } h(x,y,z) = 0)$$

$$z = 1-x-y = 1-2-1 = -2 \vee z = 1-(-2)-(-1) = 4.$$

The maximum/minimum values of f are then: $f_{max} = 4,\ f_{min} = -2.$
Note: The min/max - values of f represent the highest and lowest points on the interrsection between the plane $x+y+z=1$ and the elliptic cylinder $x^2+2y^2=6$.

Alt. 2: Use the constraint $z = 1-x-y$ to first eliminate z from the function. Subsequently, since $x^2+2y^2 = 6 \Leftrightarrow (\frac{x}{\sqrt{6}})^2 + (\frac{y}{\sqrt{3}})^2 = 1$, we set $x = \sqrt{6}\cos t,\ y = \sqrt{3}\sin t.$

This gives : $f(x,y,1-x-y) = 1-x-y = 1-\sqrt{6}\cos t - \sqrt{3}\sin t = 1-3\Big(\frac{\sqrt{6}}{3}\cos t + \frac{\sqrt{3}}{3}\sin t\Big) = 1-3\Big(\cos\delta\cos t + \sin\delta\sin t\Big) = 1-3\cos(t-\delta).$

From this expression we see directly that $f_{max} = 1-3\cdot(-1) = 4,$ and $f_{min} = 1-3\cdot 1 = -2.$

9. Calculate $\iiint\limits_K z^2\,dx\,dy\,dz\ ,\quad K = \{(x,y,z)\ \epsilon\ \mathbb{R}^3 \mid x^2+y^2+z^2 \le 2\ ,\ x^2+y^2 \ge 1\}\ .$ *(5p)*

Answer:
Alt. 1: With $r^2 = x^2+y^2$, we get $1 \le r \le \sqrt{2-z^2}$ which also means that $\sqrt{2-z^2} \ge 1 \Leftrightarrow -1 \le z \le 1$. We then get:

$$\iiint\limits_K z^2\,dx\,dy\,dz = \int_{-1}^{1} z^2\Big(\int_0^{2\pi}\int_1^{\sqrt{2-z^2}} r\,dr\,d\theta\Big)\,dz = 2\pi\int_{-1}^{1} z^2\Big[\frac{1}{2}r^2\Big]_1^{\sqrt{2-z^2}}\,dz =$$

$$\pi\int_{-1}^{1} z^2(1-z^2)\,dz = 2\pi\int_0^1 (z^2-z^4)\,dz = 2\pi\Big[\frac{1}{3}z^3 - \frac{1}{5}z^5\Big] = \frac{4\pi}{15}.$$

Alt. 2: First ‘integrating out’ z - followed by an integral over the projection of K in the xy-plane. Note that the projection is given by: $K_{xy}: 1 \le x^2+y^2 \le 2.$

$$\iiint\limits_K z^2\,dx\,dy\,dz = \int_0^{2\pi}\int_1^{\sqrt{2}}\Big(\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} z^2\,dz\Big)\,rdr\,d\theta =$$

$$\frac{4\pi}{3}\int_1^{\sqrt{2}} (2-r^2)^{\frac{3}{2}}\,r\,dr = \Big[u = 2-r^2\Big] = \frac{2\pi}{3}\int_0^1 u^{\frac{3}{2}}\,du = \frac{2\pi}{3}\Big[\frac{2}{5}u^{\frac{5}{2}}\Big]_0^1 = \frac{4\pi}{15}.$$

Alt. 3: Using spherical coordinates: $x^2+y^2 \ge 1 \Leftrightarrow \rho^2\sin^2\phi \ge 1$

$$\rho^2(1-\cos^2\phi) \ge 1 \Leftrightarrow \cos^2\phi \le \frac{\rho^2-1}{\rho^2} \Leftrightarrow -\frac{\sqrt{\rho^2-1}}{\rho} \le \cos\phi \le \frac{\sqrt{\rho^2-1}}{\rho}.$$

This condition on $\cos\phi$ also means that $\rho \ge 1$. The integral then becomes:

$$\iiint\limits_K z^2\,dx\,dy\,dz = \int_0^{2\pi}\int_{\arcsin(\frac{1}{\rho})}^{\pi-\arcsin(\frac{1}{\rho})}\int_1^{\sqrt{2}} \rho^2\cos^2\phi\,\rho^2\,d\rho\,\sin\phi\,d\phi\,d\theta =$$

$$[\cos\phi = u] = 2\pi\int_1^{\sqrt{2}}\rho^4\Big(\int_{-\frac{\sqrt{\rho^2-1}}{\rho}}^{\frac{\sqrt{\rho^2-1}}{\rho}} u^2\,du\Big)d\rho = \frac{4\pi}{3}\int_1^{\sqrt{2}}(\rho^2-1)^{\frac{3}{2}}\,\rho\,d\rho = \frac{4\pi}{15}.$$