Solutions Multivariable calculus, 2004-10-28.

1. Find an equation of the plane tangent to the surface $z = y\, e^{x^2-y} + \dfrac{x}{y}$ at the point $(1,1,2)$. *(2p)*

   Answer:
   Set: $f(x,y) = y\, e^{x^2-y} + \frac{x}{y}$. The equation is then given by:
   $z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) =$
   $2 + (2xe^{y^2-x} + \frac{1}{y})_{(1,1)}\,(x-1) + (e^{x^2-y}(1-y) - \frac{x}{y^2})_{(1,1)}\,(y-1) = 2 + 3(x-1) + (-1)(y-1) \Leftrightarrow$
   $z - 3x + y = 0.$

2. Calculate the directional derivative of $f(x,y,z) = xy + y^3 + x\ln z$ at the point $P = (2,-1,1)$ and in the direction $\boldsymbol{u} = (1,2,2)$. *(2p)*

   Answer:
   $f'_{\hat{\boldsymbol{u}}}(2,-1,1) = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} \cdot \nabla f(2,-1,1) = \frac{1}{3}\,(1,2,2) \cdot (y + \ln z, x + 3y^2, \frac{x}{z})_{(2,-1,1)} =$
   $\frac{1}{3}(1,2,2) \cdot (-1,5,2) = \frac{13}{3}.$

3. Find the following limit or show that it does not exist $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2}$. *(2p)*

   Answer:
   Set $(x,y) = (t, mt)$, $(m = const.)$. This gives: $\dfrac{xy}{2x^2+y^2} = \dfrac{mt^2}{2t^2+m^2t^2} = \dfrac{m}{m^2+2}$ $(t \neq 0)$.
   Since the expression above is m - dependent the limit can not exist. Set, e.g., $m = 1$ and $m = 3$ and confirm that the expression will approach $\frac{1}{3}$ and $\frac{3}{11}$, respectively, as $(x,y) \to (0,0)$.

4. Find all local maxima, minima, and saddle points of the function $f(x,y) = x^3 - 2y^2 + 2xy + x - 2y$. *(3p)*

   Answer:
   Critical points given by:
   $$\begin{array}{rcl} f_x &=& 0 \\ f_y &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 3x^2+2y+1 &=& 0 \\ 2x-4y-2 &=& 0 \end{array} \Leftrightarrow (x,y) = (-\tfrac{1}{3}, -\tfrac{2}{3}) \vee (x,y) = (0, -\tfrac{1}{2}).$$
   Second order derivatives:
   $$f_{xx} = \frac{\partial}{\partial x}(3x^2+2y+1) = 6x,$$
   $$f_{yy} = \frac{\partial}{\partial y}(2x-4y-2) = -4,$$
   $$f_{xy} = \frac{\partial}{\partial y}(3x^2+2y+1) = 2.$$
   Hence: $\Delta = f_{xx}\, f_{yy} - (f_{xy})^2 = -24x - 4.$

$(x,y) = (-\frac{1}{3}, -\frac{2}{3})$ gives: $\Delta = 4 > 0$. Also $f_{xx}(-\frac{1}{3}, -\frac{2}{3}) = -2 < 0$. This means that $(-\frac{1}{3}, -\frac{2}{3})$ is a local maximum point (with local maximum value $f(-\frac{1}{3}, -\frac{2}{3}) = \frac{14}{27}$).

$(x,y) = (0, -\frac{1}{2})$ gives: $\Delta = -4 < 0$. $(0, -\frac{1}{2})$ is then a saddle-point.

5. Find the absolute minimum and maximum values of $f(x,y) = x^2 + 2y^2 + 2xy - 2x - 2y$ on the set $\Delta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \,,\ \frac{x}{2} - 1 \leq y \leq -\frac{x}{2} + 1\}$. *(5p)*

Answer:
Interior critical points:

$$\begin{array}{rcl} f_x &=& 0 \\ f_y &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2x+2y-2 &=& 0 \\ 4y+2x-2 &=& 0 \end{array} \Leftrightarrow (x,y) = (1,0).$$

We note that the point $(1,0)$ lies within Δ.

Next we investigate the boundary which, in this case, can be seen to consist of the three sides of a triangle:

(a) $x = 0,\ -1 \leq y \leq 1:\ f(0,y) = 2y^2 - 2y = h_1(y) \Rightarrow h_1'(y) = 4y - 2$.
$h_1'(y) = 0 \Rightarrow y = \frac{1}{2}$.

Hence we find the following interesting (potentially extremal) points: $(0, \frac{1}{2}), (0,-1), (0,1)$, where the latter two are endpoints on the line segment.

(b) $y = \frac{x}{2} - 1,\ 0 \leq x \leq 2$:

$f(x, \frac{x}{2} - 1) = x^2 + 2(\frac{x}{2} - 1)^2 + 2x(\frac{x}{2} - 1) - 2x - 2(\frac{x}{2} - 1) = \frac{5}{2}x^2 - 7x + 4 = h_2(x) \Rightarrow$
$h_2'(x) = 5x - 7$.
$h_2'(x) = 0 \Rightarrow x = \frac{7}{5}$.

Hence two additional interesting points: $(\frac{7}{5}, -\frac{3}{10})$, $(2,0)$.

(c) $y = -\frac{x}{2} + 1,\ 0 \leq x \leq 2$:

$f(x, -\frac{x}{2} + 1) = x^2 + 2(-\frac{x}{2} + 1)^2 + 2x(-\frac{x}{2} + 1) - 2x - 2(-\frac{x}{2} + 1) = \frac{1}{2}x^2 - x = h_3(x) \Rightarrow$
$h_3'(x) = x - 1$.
$h_3'(x) = 0 \Rightarrow x = 1$.

Hence one additional interesting points: $(1, \frac{1}{2})$.

Finally, we compare the function values at these 7 points:
$f(1,0) = -1$, $f(0, \frac{1}{2}) = -\frac{1}{2}$, $f(0,-1) = 4$, $f(0,1) = 0$, $f(\frac{7}{5}, -\frac{3}{10}) = -\frac{9}{10}$, $f(2,0) = 0$, $f(1, \frac{1}{2}) = -\frac{1}{2}$. This means that $f_{min} = f(1,0) = -1$ and $f_{max} = f(0,-1) = 4$.

6. The equation $z^3 + z + x^2 + xy^3 = 2$ implicitly defines a function $z = g(x,y)$ for which $g(1,-1) = 1$. Calculate $g_{xy}(1,-1)$. *(5p)*

Answer:
Taking the partial derivative of the equation with respect to x gives:

$$3z^2 z_x + z_x + 2x + y^3 \ = \ 0 \ \Leftrightarrow \ z_x \ = \ -\frac{2x+y^3}{3z^2+1} \ \Rightarrow \ z_x(1,-1) \ = \ -\frac{2\cdot 1 + (-1)^3}{3\cdot 1 + 1} \ = \ -\frac{1}{4}.$$

Partial derivative wrt y:

$$3z^2 z_y + z_y + 3xy^2 \ = \ 0 \ \Leftrightarrow \ z_y \ = \ -\frac{3xy^2}{3z^2+1} \ \Rightarrow \ z_y(1,-1) \ = \ -\frac{3\cdot 1\cdot(-1)^2}{3\cdot 1 + 1} \ = \ -\frac{3}{4}.$$

Finally z_{xy} can be calculated from:

$$\frac{\partial}{\partial y}\Big(3z^2 z_x + z_x + 2x + y^3\Big) \ = 0 \Leftrightarrow 6zz_y z_x + 3z^2 z_{xy} + z_{xy} + 3y^2 \ = 0 \Leftrightarrow z_{xy} \ = \ -\frac{6zz_y z_x + 3y^2}{3z^2+1} \ \Rightarrow$$

$$z_{xy}(1,-1) \ = \ -\frac{33}{32}.$$

7. Calculate $\displaystyle\iint_A e^{-(4x^2+9y^2)}\, dx\, dy \ , \quad A \ = \ \{(x,y) \in \mathbb{R}^2 |\ y \geq 0,\ 4x^2 + 9y^2 \leq 1\}$. *(3p)*

Answer:

Set $x = \dfrac{1}{2}\, r\, \cos\theta,\ \ y = \dfrac{1}{3}\, r\, \sin\theta$. Then $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{r}{6}$.

The limits in the polar coordinates (consistent with the conditions on A) are:

$0 \leq \theta \leq \pi,\ \ 0 \leq r \leq 1$.

$$\text{Inserted :} \iint_A e^{-(4x^2+9y^2)}\, dx\, dy \ = \ \int_0^{\pi}\int_0^{1} e^{-r^2}\, \frac{r}{6}\, dr\, d\theta \ = \ \frac{\pi}{6}\left[-\frac{1}{2}e^{-r^2}\right]_0^1 \ = \ \frac{\pi}{12}\left(1 - e^{-1}\right) .$$

8. Calculate $\displaystyle\iint_D \frac{1}{x^2}\, \ln\Big(\frac{y}{x}\Big)\, dx\, dy \ , \quad D \ = \ \{(x,y) \in \mathbb{R}^2 |\ 1 \leq x + y \leq 2,\ x \leq y \leq 2x\}$. *(3p)*

Hint: Use the following transformation $u \ = \ x + y, \quad v \ = \ \dfrac{y}{x}$.

Answer:

Using the hint we obtain $\dfrac{\partial(x,y)}{\partial(u,v)} \ = \ \left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1} \ = \ \dfrac{x^2}{x+y}$.

$$\text{Inserted:} \ \iint_D \frac{1}{x^2}\, \ln\Big(\frac{y}{x}\Big)\, dx\, dy \ = \ \int_1^2\int_1^2 \frac{1}{u}\, \ln v\, du\, dv \ = \ \Big[\ln u\Big]_1^2 \Big[v \ln v - v\Big]_1^2 \ = \ (2\ln 2 - 1)\ln 2.$$

9. Calculate $\iiint\limits_K z\,dx\,dy\,dz$, $K = \{(x,y,z) \in \mathbb{R}^3 \,|\, 1 \le x^2+y^2+z^2 \le 4,\ z \ge \sqrt{x^2+y^2}\}$. *(5p)*

Answer:
Using spherical coordinates the condition $z \ge \sqrt{x^2+y^2}$ translates to $\rho\cos\phi \ge \rho\sin\phi \Rightarrow 0 \le \phi \le \frac{\pi}{4}$. Likewise $1 \le x^2+y^2+z^2 \le 4$ means that $1 \le \rho \le 2$. Inserted:

$$\iiint\limits_K z\,dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\rho=1}^{2} \rho\cos\phi\,\rho^2\,\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\frac{\pi}{4}} \cos\phi\,\sin\phi\,d\phi\int_1^2 \rho^3\,d\rho =$$

$$[u=\cos\phi] = 2\pi\int_0^{\frac{\sqrt{2}}{2}} u\,du\left[\frac{1}{4}\rho^4\right]_1^2 = 2\pi\cdot\frac{1}{4}\cdot\frac{15}{4} = \frac{15\pi}{8}\,.$$

Alternatively, we may perform the integral using cylindrical coordinates.
Since $\sqrt{1-x^2-y^2} > \sqrt{x^2+y^2}$ when $r = \sqrt{x^2+y^2} < \frac{\sqrt{2}}{2}$, the integral will in this case have two separate contributions:

$$\iiint\limits_K z\,dx\,dy\,dz = 2\pi\int_0^{\frac{\sqrt{2}}{2}}\int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} z\,dz\,r\,dr + 2\pi\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} z\,dz\,r\,dr =$$

$$3\pi\int_0^{\frac{\sqrt{2}}{2}} r\,dr + 2\pi\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} (2r-r^3)\,dr = \frac{15\pi}{8}.$$