Solutions multivariable calculus, 2005-10-27.

1. Find an equation of the plane tangent to the surface $\sqrt{x^2+y^2} - z = 1$ at the point $(3,-4,4)$. *(2p)*

   Answer:
   Set: $f(x,y) = \sqrt{x^2+y^2} - 1$. The equation of the plane tangent is then given by:
   $z = f(3,-4) + f_x(3,-4)(x-3) + f_y(3,-4)(y-(-4)) =$
   $4 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)_{(3,-4)}(x-3) + \left(\frac{y}{\sqrt{x^2+y^2}}\right)_{(3,-4)}(y+4) = 4 + \frac{3}{5}(x+3) + (-\frac{4}{5}(y+4) \Leftrightarrow$
   $3x - 4y - 5z - 5 = 0.$

2. Find the following limit or show that it does not exist $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. *(2p)*

   Answer:
   Compare the results of the following 'routes' towards $(0,0)$:
   $(x,y) = (t,0)$ gives : $\frac{x^2-y^2}{x^2+y^2} = \frac{t^2}{t^2} = 1 \ (t \neq 0)$,
   while $(x,y) = (0,t)$ gives : $\frac{x^2-y^2}{x^2+y^2} = \frac{-t^2}{t^2} = -1 \ (t \neq 0)$.
   These two different results indicate that the limit does not exist.

3. Calculate the directional derivative of $f(x,y,z) = \ln(x^2-y) + ye^{-z}$ at the point $P = (3,1,0)$ and in the direction $\boldsymbol{v} = (2,2,1)$. *(2p)*

   Answer:
   $f'_{\hat{\boldsymbol{v}}}(3,1,0) = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla f(3,1,0) = \frac{1}{3}(2,2,1) \cdot (\frac{2x}{x^2-y}, -\frac{1}{x^2-y} + e^{-z}, -ye^{-z})_{(3,1,0)} =$
   $\frac{1}{3}(2,2,1) \cdot (\frac{3}{4}, \frac{7}{8}, -1) = \frac{3}{4}.$

4. Find the (absolute) maximum and minimum values of $f(x,y) = 2x^2 + y^2 - 8x$ subject to the constraint $x^2 + y^2 = 6x$. *(3p)*

   Answer:
   Alt. 1: Using the method of Lagrange multipliers (let $g(x,y) = x^2+y^2-6x = 0$ represent the constraint):
   $$\begin{array}{rcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \end{array} \Leftrightarrow \begin{array}{rcl} 4x-8 &=& \lambda(2x-6) \\ 2y &=& 2\lambda y \end{array} \Leftrightarrow \begin{array}{rcl} 4x-8 &=& \lambda(2x-6) \\ y &=& 0 \vee \lambda = 1 \end{array}$$
   $y = 0$ into the constraint gives: $x^2 - 6x = 0 \Leftrightarrow x = 6 \vee x = 0 \Rightarrow (x,y) = (0,0) \vee (x,y) = (6,0)$.

   $\lambda = 1$ gives $4x - 8 = 2x - 6 \Leftrightarrow x = 1$, which, when inserted into the constraint gives $y^2 = 5 \Leftrightarrow y = \pm\sqrt{5} \Rightarrow (x,y) = (1, \pm\sqrt{5})$.
   We then calculate the function values: $f(0,0) = 0$, $f(6,0) = 24$, $f(1,\pm\sqrt{5}) = -1$.

Conclusion : $f_{min} = -1, \; f_{max} = 24.$

Alt. 2: Use the constraint to eliminate y^2 from the function.
This gives : $f(x, \pm\sqrt{6x - x^2}) = x^2 - 2x = h(x)$. Note that h is only defined on the interval $0 \le x \le 6$. Min/max - values of the single-variable function h are then obtained either at the endpoints $x = 0, x = 6$ or at solutions of $h'(x) = 0 \;\Leftrightarrow\; x = 1$. This gives $h(0) = f(0,0) = 0,\; h(6) = f(6,0) = 24,\; h(1) = f(1, \pm\sqrt{5}) = -1.$

(A third alternative would be to parametrize the circle $x^2 + y^2 - 6x = 0$).

5. Calculate $\displaystyle\iint_A \frac{\sin x}{x}\, dx\, dy$, where the region A is a triangle with corners at the points $(0,0), (\pi, 0)$, and (π, π). *(3p)*

Answer:
Note that the integral can be performed analytically only by an initial integration in y followed by integration in x (reversing the order of integration will require a numerical computation):

$$\iint_A \frac{\sin x}{x}\, dx\, dy \;=\; \int_0^\pi \frac{\sin x}{x}\Big(\int_0^x dy\Big)\, dx \;=\; \int_0^\pi \sin x\, dx \;=\; \Big[-\cos x\Big]_0^\pi \;=\; 2.$$

6. Calculate $\displaystyle\iint_D xy\, dx\, dy$, where

$D \;=\; \{(x,y) \in \mathbb{R}^2 |\; 2 \le xy \le 4\,,\;\; 1 \le \frac{y}{x} \le 2,\; x > 0\}$. *(3p)*

Answer:
The region D in the $xy-$plane can be transformed into a rectangle D_{uv} by the following transformation:

$u = xy,\; v = \dfrac{y}{x}$, which gives $2 \le u \le 4,\; 1 \le v \le 2$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \Big(\dfrac{\partial(u,v)}{\partial(x,y)}\Big)^{-1} = \dfrac{x}{2y} = \dfrac{1}{2v}.$

Inserted into the integral:

$$\iint_D xy\, dx\, dy \;=\; \frac{1}{2}\int_2^4\int_1^2 u\,\frac{1}{v}\, dv\, du \;=\; \frac{1}{2}\Big[\frac{1}{2}u^2\Big]_2^4 \Big[\ln v\Big]_1^2 \;=\; 3\ln 2.$$

7. Find the (absolute) minimum and maximum values of $g(x,y) \;=\; xy \;+\; x \;+\; 2y$ on the set $\Delta \;=\; \{(x,y) \in \mathbb{R}^2 \;|\; x^2 \le y \le 2 - x\}$. *(5p)*

Answer:
Interior critical points:

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \;\Leftrightarrow\; \begin{matrix} y + 1 = 0 \\ x + 2 = 0 \end{matrix} \;\Leftrightarrow\; (x,y) = (-2,-1).$$

This point clearly lies outside Δ.

Next we investigate the boundary which, in this case, can be seen to consist of two curve segments with common endpoints at the solutions of $x^2 = 2 - x \Leftrightarrow x = -2 \vee x = 1$.

(a) $y = x^2,\ -2 \le x \le 1:\ f(x, x^2) = x^3 + 2x^2 + x = h_1(x) \Rightarrow h_1'(x) = 3x^2 + 4x + 1$.
$h_1'(x) = 0 \Rightarrow x = -1 \vee x = -\frac{1}{3}$.

This gives the following interesting points: $(-2, 4)$, $(-1, 1)$, $(-\frac{1}{3}, \frac{1}{9})$, $(1, 1)$ (the first and last points are endpoints).

(b) $y = 2 - x,\ -2 \le x \le 1$:

$f(x, 2 - x) = -x^2 + x + 4 = h_2(x) \Rightarrow h_2'(x) = -2x + 1$.
$h_2'(x) = 0 \Rightarrow x = \frac{1}{2}$.

This gives a single additional interesting points (the endpoints of the segment are already accounted for) : $(\frac{1}{2}, \frac{3}{2})$.

Finally, we compare the function values at these 5 points:
$f(-2, 4) = -2,\ f(-1, 1) = 0,\ f(-\frac{1}{3}, \frac{1}{9}) = -\frac{4}{27},\ f(\frac{1}{2}, \frac{3}{2}) = \frac{17}{4},\ f(1, 1) = 4$.

Conclusion: $f_{min} = -2,\ f_{max} = \dfrac{17}{4}$.

8. Calculate $\displaystyle\iiint_K (x^2 + y^2)\, dx\, dy\, dz$,

where $K = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{\dfrac{x^2 + y^2}{3}} \le z \le \sqrt{1 - x^2 - y^2}\}$. *(5p)*

Answer:

Using spherical coordinates the condition $z \ge \sqrt{\dfrac{x^2 + y^2}{3}}$ translates to

$\rho \cos\phi \ge \dfrac{1}{\sqrt{3}} \rho \sin\phi \Rightarrow 0 \le \phi \le \dfrac{\pi}{3}$. Furthermore $z \le \sqrt{1 - x^2 - y^2}$ means that $0 \le \rho \le 1$.
Inserted:

$$\iiint_K (x^2 + y^2)\, dx\, dy\, dz = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=0}^{1} \rho^2 \sin^2\phi\, \rho^2\ \sin\phi\, d\rho\, d\phi\, d\theta =$$

$$2\pi \int_0^{\frac{\pi}{3}} (1 - \cos^2\phi) \sin\phi d\phi \int_0^1 \rho^4 d\rho = [u = \cos\phi] = 2\pi \int_{\frac{1}{2}}^1 (1 - u^2) du \cdot \left[\frac{1}{5}\rho^5\right]_0^1 = 2\pi \cdot \frac{5}{24} \cdot \frac{1}{5} = \frac{\pi}{12} .$$

Alternatively, using plane polar coordinates:

Note that the projection K_{xy} on to the xy-plane is given by $\sqrt{\dfrac{x^2 + y^2}{3}} \le \sqrt{1 - x^2 - y^2} \Leftrightarrow$
$r = \sqrt{x^2 + y^2} \le \dfrac{\sqrt{3}}{2}$. This gives:

$$\iiint\limits_{K} (x^2+y^2)\,dx\,dy\,dz = \iint\limits_{K_{xy}} (x^2+y^2)\Big(\int\limits_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{1-x^2-y^2}} dz\Big)\,dx\,dy = 2\pi \int\limits_{0}^{\frac{\sqrt{3}}{2}} \Big(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\Big) r^2\, r\,dr =$$

$$[u = 1-r^2] = \pi \int\limits_{\frac{1}{4}}^{1} \sqrt{u}(1-u)\,du - \frac{2\pi}{\sqrt{3}} \int\limits_{0}^{\frac{\sqrt{3}}{2}} r^4\,dr = \frac{47\pi}{240} - \frac{9\pi}{80} = \frac{\pi}{12}.$$

9. Calculate $\iint\limits_{S} xy\sqrt{5-4z}\,dS$,

where $S = \{(x,y,z) \in \mathbb{R}^3 \mid z = 1-x^2-y^2,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$. *(5p)*

Answer:

Note that when $z = 1-x^2-y^2$ the integrand will be $xy\sqrt{5-4z} = xy\sqrt{1+4x^2+4y^2}$. The angular and radial limits for the integration over S_{xy} will then be $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. This gives:

$$\iint\limits_{S} xy\sqrt{5-4z}\,dS = \iint\limits_{S_{xy}} xy\Big(1+4x^2+4y^2\Big)^{\frac{1}{2}} \Big(1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2\Big)^{\frac{1}{2}} dx\,dy =$$

$$\int\limits_{0}^{\frac{\pi}{2}} \int\limits_{0}^{1} r\cos\theta\, r\sin\theta\, (1+4r^2)^{\frac{1}{2}}\, (1+4r^2)^{\frac{1}{2}}\, r dr\, d\theta = \int\limits_{0}^{\frac{\pi}{2}} \cos\theta\, \sin\theta\, d\theta \int\limits_{0}^{1} r^3(1+4r^2)\,dr =$$

$$\Big[-\frac{1}{2}\cos^2\theta\Big]_0^{\frac{\pi}{2}} \Big[\frac{1}{4}r^4 + \frac{4}{6}r^6\Big]_0^1 = \frac{11}{24}.$$