Solutions to exam 2003-10-31.

1. The directional derivative :

$$\begin{aligned} g'_{\mathbf{v}}(1,1,1) &= \frac{\mathbf{v}}{|\mathbf{v}|}\cdot\nabla g(1,1,1) = \frac{(2,1,2)}{|(2,1,2)|}\cdot(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})_{(1,1,1)} = \\ &= \frac{1}{3}(2,1,2)\cdot(2yx^{2y-1},2x^{2y}\ln x+z,y+2z)_{(1,1,1)} = \\ &= \frac{1}{3}(2,1,2)\cdot(2,1,3) = \frac{11}{3} \end{aligned}$$

2. A normal vector to the tangent plane is given by:

$$\mathbf{n} = \nabla(xy^2+yz-z^3+2)_{(-2,1,1)} = (y^2,2xy+z,y-3z^2)_{(-2,1,1)} = (1,-3,-2)$$

The equation for the tangent plane is then given by:

$$\mathbf{n}\cdot(x-(-2),y-1,z-1)=0 \Leftrightarrow x-3y-2z+7=0$$

3. Using (plane) polar coordinates we get:

$$\frac{xy^2-x^3}{x^2+y^2} = \frac{r^3(\cos\theta\sin^2\theta-\cos^3\theta)}{r^2} = r(\cos\theta\sin^2\theta-\cos^3\theta)$$

Now, the angular part of the expression above is bounded, and we must have $r\to 0$ as $(x,y)\to(0,0)$.The limit is then:

$$\lim_{(x,y)\to(0,0)}\frac{xy^2-x^3}{x^2+y^2}=0$$

4. Set:$x=h, y=1+k$.This gives:

$$\begin{aligned} \ln(2h^2+1+k) &= 2h^2+k-\frac{1}{2}(2h^2+k)^2+\frac{1}{3}(2h^2+k)^3+O(|(h,k)|^4) = \\ & k+2h^2-\frac{1}{2}k^2+\frac{1}{3}k^3-2h^2k+O(|(h,k)|^4) \end{aligned}$$

The Taylor-polynomial is therefore:

$$p_3(h,k) = k+2h^2-\frac{1}{2}k^2+\frac{1}{3}k^3-2h^2k = (y-1)+2x^2-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-2x^2(y-1)$$

5. Using the suggested transformation we obtain:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}+\frac{1}{y}\frac{\partial f}{\partial v}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = -\frac{x}{y^2}\frac{\partial f}{\partial v}$$

Inserted ino the original PDE this gives:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial u} = 2x^2 \Leftrightarrow \frac{\partial f}{\partial u} = 2u \Leftrightarrow f = u^2+g(v) \Leftrightarrow f(x,y) = x^2+g(\frac{x}{y})$$

where g is an arbitrary (differentiable) function of one variable. The particular solution is obtained by:

$$f(1,y) = 1+\sqrt{y+1} \Leftrightarrow 1^2+g(\frac{1}{y}) = 1+\sqrt{y+1} \Leftrightarrow (t=\frac{1}{y}), g(t) = \sqrt{\frac{1}{t}+1}$$

Subsequently insert this expression for g into the general solution, and we obtain:

$$f(x,y) = x^2+\sqrt{\frac{y}{x}+1} = x^2+\sqrt{\frac{x+y}{x}}$$

6. The set Δ is a rectangle which contains its own boundary.Since f is a continuous function which is also defined on a compact set we then know that the function has absolute maximum/minimum values on said set. These values can either be found on the boundary of Δ or at an interior critical point. We proceed to first find all (possible) critical points:

$$\begin{array}{c} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{array} \Leftrightarrow \begin{array}{c} 2x + y - 6 = 0 \\ 2y + x = 0 \end{array} \Leftrightarrow \begin{array}{c} x = 4 \\ y = -2 \end{array}$$

The only interior critical point is therefore $P_1 = (4, -2)$. The 4 parts of the boundary are then investigated:

$$y = 0, 0 \leq x \leq 5$$

gives

$$f(x, 0) = x^2 - 6x = h_1(x), h_1'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$$

This gives three new interesting points: $P_2 = (0, 0), P_3 = (3, 0), P_4 = (5, 0)$

$$y = -3, 0 \leq x \leq 5$$

gives :

$$f(x, -3) = x^2 - 9x + 9 = h_2(x), h_2'(x) = 0 \Rightarrow 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$$

which yields three additional points: $P_5 = (0, -3), P_6 = (\frac{9}{2}, -3), P_7 = (5, -3)$.

$$x = 0, -3 \leq y \leq 0$$

gives

$$f(0, y) = y^2$$

This is (obviously) an increasing function. The line-segment therefore does not yield any new points, since the end-points are already accounted for.

$$x = 5, -3 \leq y \leq 0$$

gives

$$f(5, y) = y^2 + 5y - 5 = h_3(y), h_3'(y) = 0 \Rightarrow 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$$

which yields the final point:$P_8 = (5, -\frac{5}{2})$.The largest/smallest values values of the function are then to be found at one of these points:

$$f(4, -2) = -12, f(0, 0) = 0, f(3, 0) = -9, f(5, 0) = -5, f(0, -3) = 9, f(\frac{9}{2}, -3) = -\frac{45}{4}, f(5, -3) = -11$$

The conclusion is then:

$$f_{\min} = -12, f_{\max} = 9$$

7. The given limits in y means that

$$x^2 \leq 2 - x^2 \Rightarrow 0 \leq x \leq 1$$

(since x is positive).The integral can then be written as:

$$\begin{aligned} \int_0^1 \sqrt{x} \left(\int_{x^2}^{2-x^2} y dy \right) dx &= \int_0^1 \sqrt{x} \left[\frac{1}{2} y^2 \right]_{x^2}^{2-x^2} dx = \frac{1}{2} \int_0^1 \left(4x^{1/2} - 4x^{5/2} \right) dx = \\ 2 \left[\frac{2}{3} x^{3/2} - \frac{2}{7} x^{7/2} \right]_0^1 &= \frac{16}{21} \end{aligned}$$

8. The integral can be written as:

$$\begin{aligned}\int_0^1 y^2 \left(\int_0^y e^{xy} dx\right) dy &= \int_0^1 y^2 \left[\frac{1}{y} e^{xy}\right]_0^y dy = \int_0^1 \left(ye^{y^2} - y\right) dy = \\ \left[\frac{1}{2} e^{y^2} - \frac{1}{2} y^2\right]_0^1 &= \frac{1}{2} e - 1\end{aligned}$$

Note that the integral has to be iterated in precisely this order (x first, then y).Otherwise, we will have to contend with a very unpleasant final x-integration.

9. The limits in the xy-plane are given by $x \geq 0, 0 \leq y \leq x$ and

$$\sqrt{4 - x^2 - y^2} \geq 1 \Leftrightarrow x^2 + y^2 \leq 3$$

This would suggest an integration in the z-direction followed by a change of variables to plane polar coordinates:

$$\sqrt{x^2 + y^2} = r \leq \sqrt{3}$$

$$r \cos\varphi \geq 0, 0 \leq r \sin\varphi \leq r \cos\varphi \Rightarrow 0 \leq \varphi \leq \frac{\pi}{4}$$

$$\begin{aligned}\int\int\int_K zy^2 dxdydz &= \int\int_{K_{xy}} y^2 \left(\int_1^{\sqrt{4-x^2-y^2}} zdz\right) dxdy = \\ \int\int_{K_{xy}} y^2 \left[\frac{1}{2} z^2\right]_1^{\sqrt{4-x^2-y^2}} dxdy &= \\ \frac{1}{2}\int\int_{K_{xy}} y^2 \left(3 - x^2 - y^2\right) dxdy &= \frac{1}{2}\int_0^{\sqrt{3}} \left(3 - r^2\right) \left(\int_0^{\frac{\pi}{4}} r^2 \sin^2\varphi d\varphi\right) rdr = \\ \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\varphi d\varphi \int_0^{\sqrt{3}} r^3 \left(3 - r^2\right) dr &= \frac{1}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right)\frac{9}{4} = \frac{9}{64}(\pi - 2)\end{aligned}$$

The integral could, alternatively, be calculated by an initial integration over K_z,that is, the intersection of the plane $z = a, 1 \leq a \leq 2$ and the body K.Subsequently, we integrate in the z-direction. A third option is to use spherical coordinates.