Solutions for Test-Exam.

1. $$\lim_{x\to\infty} \frac{1}{\sqrt{x^2+3x}-x} = \lim_{x\to\infty} \frac{\sqrt{x^2+3x}+x}{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)} =$$
$$\lim_{x\to\infty} \frac{\sqrt{x^2+3x}+x}{3x} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{3}{x}}+1}{3} = \frac{2}{3}.$$

2. The equation of the tangent: $y - \arctan(1^3) = D(\arctan(x^3))_{x=1}\,(x - 1) \Rightarrow$
$$y - \frac{\pi}{4} = \Big(\frac{3x^2}{1+x^6}\Big)_{x=1}(x - 1) \Rightarrow y = \frac{3}{2}\,x + \frac{\pi}{4} - \frac{3}{2}.$$

3. a) $\dfrac{5x+9}{x^2+4x+3} = \dfrac{5x+9}{(x+1)(x+3)} = \dfrac{2}{x+1} + \dfrac{3}{x+3}$. Inserted:
$$\int_1^3 \frac{2}{x+1}\,dx + \int_1^3 \frac{3}{x+3}\,dx = 2\,\ln(x+1)|_1^3 + 3\,\ln(x+3)|_1^3 = 3\ln 3 - \ln 2.$$
b) Set: $u(x) = x^3+1 \Rightarrow du = 3x^2\,dx,\ u(0) = 1,\ u(2) = 9.$ Inserted:
$$\int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_1^9 \sqrt{u}\,du = \frac{1}{3}\,\frac{2}{3}\,u^{3/2}|_1^9 = \frac{52}{9}.$$
c) Set: $u(x) = \sin x \Rightarrow du = \cos x dx,\ u(0) = 0, u(\frac{\pi}{2}) = 1$. Inserted:
$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x+1}\,dx = \int_0^1 \frac{1}{u^2+1}\,du = \arctan u|_0^1 = \frac{\pi}{4}\ .$$
d) $$\int x\,(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2\,\frac{1}{x}\,2\,\ln x\,dx =$$
$$\frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \int \frac{1}{2}x^2\,\frac{1}{x}\,dx =$$
$$\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C\ .$$

4. The triangle is right-angled with base x and height e^{-2x^2}. The area is thus $f(x) = \frac{1}{2}xe^{-2x^2}$. Taking the derivative of f:

$D(f(x)) = \frac{1}{2}\,(e^{-2x^2} + x(-4x)e^{-2x^2}) = \frac{1}{2}e^{-2x^2}\,(1+2x)(1-2x)$. From this expression we see that f is increasing in the interval $0 \le x \le \frac{1}{2}$ and decreasing when $x \ge \frac{1}{2}$. This means that the largest value is obtained when $x = \frac{1}{2}$:
$$f_{max} = f(\frac{1}{2}) = \frac{1}{2}\,\frac{1}{2}e^{-2(1/2)^2} = \frac{1}{4}\,e^{-1/2} = \frac{1}{4\sqrt{e}}.$$

5. Zeroes: $f(x) \;=\; 0 \;\Leftrightarrow\; x^2 - x \;=\; 0 \Leftrightarrow\; x = 0 \vee x = 1$.

   Local extrema:

   $$D(f(x)) = \frac{(2x-1)(x-2) \;-\; (x^2-x)}{(x-2)^2} = \frac{x^2-4x+2}{(x-2)^2} = \frac{(x-(2+\sqrt{2}))(x-(2-\sqrt{2}))}{(x-2)^2}.$$

   Local maximum for $x = 2-\sqrt{2}$, local minimum for $x = 2+\sqrt{2}$:

   $f_{max} \;=\; f(2-\sqrt{2}) \;=\; 3-\sqrt{2},\;\; f_{min} \;=\; f(2+\sqrt{2}) \;=\; 3+\sqrt{2}.$

   Asymptotes: Vertical asymptote $x = 2$ since $f(x) \to \pm\infty$ when $x \to 2$.

   Polynomial division gives: $f(x) \;=\; x+1+\dfrac{2}{x-2}$. This means that the function has the (oblique)asymptote $y = x+1$ since $\displaystyle\lim_{x\to\pm\infty} \frac{2}{x-1} \;=\; 0.$

6. The body has a cross-section area given by :

   $A(x) \;=\; \pi\;(y(x))^2 \;=\; \pi\;(e^{x^2}\sqrt{x})^2 \;=\; \pi\; e^{2x^2}\; x.$

   This gives the volume :

   $$\int_0^1 A(x)\,dx \;=\; \pi\int_0^1 e^{2x^2}\,x\,dx \;=\; (u(x) = x^2) \;=\; \frac{\pi}{2}\int_0^1 e^{2u}\,du \;=\; \frac{\pi}{2}\,\frac{1}{2}\Big[e^{2u}\Big]_0^1 \;=\; \frac{\pi}{4}\,(e^2-1)\,.$$