

Solutions for Test-Exam.

$$1. \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 3x} - x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3x} + x}{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)} =$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3x} + x}{3x} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{3}{x}} + 1}{3} = \frac{2}{3}.$$

2. The equation of the tangent: $y - \arctan(1^3) = D(\arctan(x^3))_{x=1} (x - 1) \Rightarrow$

$$y - \frac{\pi}{4} = \left(\frac{3x^2}{1+x^6} \right)_{x=1} (x - 1) \Rightarrow y = \frac{3}{2}x + \frac{\pi}{4} - \frac{3}{2}.$$

3. a) $\frac{5x+9}{x^2+4x+3} = \frac{5x+9}{(x+1)(x+3)} = \frac{2}{x+1} + \frac{3}{x+3}$. Inserted:

$$\int_1^3 \frac{2}{x+1} dx + \int_1^3 \frac{3}{x+3} dx = 2 \ln(x+1)|_1^3 + 3 \ln(x+3)|_1^3 = 3 \ln 3 - \ln 2.$$

b) Set: $u(x) = x^3 + 1 \Rightarrow du = 3x^2 dx$, $u(0) = 1$, $u(2) = 9$. Inserted:

$$\int_0^2 x^2 \sqrt{x^3 + 1} dx = \frac{1}{3} \int_1^9 \sqrt{u} du = \frac{1}{3} \frac{2}{3} u^{3/2} \Big|_1^9 = \frac{52}{9}.$$

c) Set: $u(x) = \sin x \Rightarrow du = \cos x dx$, $u(0) = 0$, $u(\frac{\pi}{2}) = 1$. Inserted:

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x + 1} dx = \int_0^1 \frac{1}{u^2 + 1} du = \arctan u \Big|_0^1 = \frac{\pi}{4}.$$

d) $\int x (\ln x)^2 dx = \frac{1}{2} x^2 (\ln x)^2 - \int \frac{1}{2} x^2 \frac{1}{x} 2 \ln x dx =$

$$\frac{1}{2} x^2 (\ln x)^2 - \int x \ln x dx = \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{2} x^2 \ln x + \int \frac{1}{2} x^2 \frac{1}{x} dx =$$

$$\frac{1}{2} x^2 (\ln x)^2 - \frac{1}{2} x^2 \ln x + \frac{1}{4} x^2 + C.$$

4. The triangle is right-angled with base x and height e^{-2x^2} . The area is thus $f(x) = \frac{1}{2} x e^{-2x^2}$. Taking the derivative of f :

$D(f(x)) = \frac{1}{2} (e^{-2x^2} + x(-4x)e^{-2x^2}) = \frac{1}{2} e^{-2x^2} (1 + 2x)(1 - 2x)$. From this expression we see that f is increasing in the interval $0 \leq x \leq \frac{1}{2}$ and decreasing when $x \geq \frac{1}{2}$. This means that the largest value is obtained when $x = \frac{1}{2}$:

$$f_{max} = f\left(\frac{1}{2}\right) = \frac{1}{2} \frac{1}{2} e^{-2(1/2)^2} = \frac{1}{4} e^{-1/2} = \frac{1}{4\sqrt{e}}.$$

5. Zeroes: $f(x) = 0 \Leftrightarrow x^2 - x = 0 \Leftrightarrow x = 0 \vee x = 1$.

Local extrema:

$$D(f(x)) = \frac{(2x-1)(x-2) - (x^2-x)}{(x-2)^2} = \frac{x^2-4x+2}{(x-2)^2} = \frac{(x-(2+\sqrt{2}))(x-(2-\sqrt{2}))}{(x-2)^2}.$$

Local maximum for $x = 2 - \sqrt{2}$, local minimum for $x = 2 + \sqrt{2}$:

$$f_{max} = f(2 - \sqrt{2}) = 3 - \sqrt{2}, \quad f_{min} = f(2 + \sqrt{2}) = 3 + \sqrt{2}.$$

Asymptotes: Vertical asymptote $x = 2$ since $f(x) \rightarrow \pm\infty$ when $x \rightarrow 2$.

Polynomial division gives: $f(x) = x + 1 + \frac{2}{x-2}$. This means that the function has the (oblique) asymptote $y = x + 1$ since $\lim_{x \rightarrow \pm\infty} \frac{2}{x-2} = 0$.

6. The body has a cross-section area given by :

$$A(x) = \pi (y(x))^2 = \pi (e^{x^2} \sqrt{x})^2 = \pi e^{2x^2} x.$$

This gives the volume :

$$\int_0^1 A(x) dx = \pi \int_0^1 e^{2x^2} x dx = (u(x) = x^2) = \frac{\pi}{2} \int_0^1 e^{2u} du = \frac{\pi}{2} \frac{1}{2} [e^{2u}]_0^1 = \frac{\pi}{4} (e^2 - 1).$$