Seminar 4: Pole-placement design

February 10, 2006

Outline

1. Feedback design
   - The polynomial equation
   - Pre-chosen factors
   - Pole-placement versus IMC
2. Design trade-off
   - Avoid model matching formulation
   - Speed versus noise sensitivity
3. Example: Damping of a pendulum
   - Cascade feedback design
   - Reference feedforward design

Feedback can change stability properties

Process

$$Ay = Bu$$

Characteristic polynomial A determines stability

Feedback controller

$$Ru = -Sy$$

Closed-loop system

$$ARy = BRu = -BSy \rightarrow \underbrace{(AR + BS)}_{A_c} y = 0$$

Closed-loop characteristic polynomial A_c determines stability

The pole-placement problem

Given process model polynomials A and B
Given (chosen) closed-loop poles $\lambda_1, \ldots, \lambda_n$

$$\rightarrow A_c(\mathrm{q}^{-1}) = (1-\lambda_1\mathrm{q}^{-1})\ldots(1-\lambda_n\mathrm{q}^{-1}) = 1+a_{c_1}\mathrm{q}^{-1}+\ldots+a_{c_n}\mathrm{q}^{-n}$$

Calculate controller polynomials R and S

$$AR + BS = A_c \quad \rightarrow \begin{cases} R \\ S \end{cases}$$

Cancel common factors

Assume A and B coprime (no common factors)
Study common factors between A_c and A or B

$$(A_1 A_2)R + (B_1 B_2)S = A_{c1} A_{c2}$$

Same factors must be on left and right hand side:

- $A_{c2} = A_2 \Rightarrow S = S_1 A_2$. Cancel common factor A_2

$$A_1 R + B S_1 = A_{c1} \rightarrow \begin{cases} R \\ S_1 \end{cases}$$

- $A_{c2} = B_2 \Rightarrow R = R_1 B_2$

$$A R_1 + B_1 S = A_{c1} \rightarrow \begin{cases} R_1 \\ S \end{cases}$$

An equivalent equation system

Assume A, B and A_c coprime
Identify coefficient of q^{-k}:

$$\sum_{i+j=k} a_i r_j + \sum_{i+j=k} b_i s_j = a_{c_k}, \qquad k = 0, \ldots, n$$

In matrix form

$$n+1 \left\{ \underbrace{\begin{pmatrix} a_0 & \ldots & 0 \\ a_1 & \ddots & \vdots \\ a_2 & \ddots & a_0 \\ \vdots & \ddots & \vdots \end{pmatrix}}_{\deg R+1} \underbrace{\begin{pmatrix} b_0 & \ldots & 0 \\ b_1 & \ddots & \vdots \\ b_2 & \ddots & b_0 \\ \vdots & \ddots & \vdots \end{pmatrix}}_{\deg S+1} \right. \begin{pmatrix} r_0 \\ \vdots \\ s_0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{c_0} \\ a_{c_1} \\ \vdots \\ a_{c_n} \end{pmatrix}$$

Unique solution if $n+1 = \deg R + \deg S + 2$

Degree conditions for unique solution

To solve

$$AR + BS = A_c \rightarrow \begin{cases} R \\ S \end{cases}$$

choose

$$\begin{cases} \deg A_c \leq \deg A + \deg B - 1 \\ \deg R = \deg B - 1 \\ \deg S = \deg A - 1 \end{cases}$$

Example

Process model

$$\frac{B}{A} = \frac{b_1 \mathrm{q}^{-1} + b_2 \mathrm{q}^{-2}}{1 + a_1 \mathrm{q}^{-1} + a_2 \mathrm{q}^{-2}}$$

Degree conditions give

$$\begin{array}{ll} \deg A_c \leq \deg AB - 1 = 3, & A_c = 1 + a_{c_1}\mathrm{q}^{-1} + a_{c_2}\mathrm{q}^{-2} + a_{c_3}\mathrm{q}^{-3} \\ \deg R = \deg B - 1 = 1, & R = r_0 + r_1 \mathrm{q}^{-1} \\ \deg S = \deg A - 1 = 1, & S = s_0 + s_1 \mathrm{q}^{-1} \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_2 & a_1 & b_2 & b_1 \\ 0 & a_2 & 0 & b_2 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{c_1} \\ a_{c_2} \\ a_{c_3} \end{pmatrix} \quad \rightarrow \begin{pmatrix} r_0 \\ r_1 \\ s_0 \\ s_2 \end{pmatrix}$$

Example: simplifications

Use that $r_0 = 1$ and eliminate first row and column

$$\begin{pmatrix} 1 & b_1 & 0 \\ a_1 & b_2 & b_1 \\ a_2 & 0 & b_2 \end{pmatrix} \begin{pmatrix} r_1 \\ s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} a_{c_1} - a_1 \\ a_{c_1} - a_2 \\ a_{c_3} \end{pmatrix} \quad \rightarrow \begin{pmatrix} r_1 \\ s_0 \\ s_2 \end{pmatrix}$$

Alternatively:
Evaluate at roots $A(p_1) = A(p_2) = B(z_1) = 0$

$$\left.\begin{array}{r} B(p_1)S(p_1) = A_c(p_1) \\ B(p_2)S(p_2) = A_c(p_2) \end{array}\right\} \rightarrow \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$$

$$A(z_1)R(z_1) = A_c(z_1) \qquad \rightarrow r_1$$

Pre-chosen fixed factors in R and S

Closed-loop disturbance response $d \rightarrow y$

$$y = \frac{RC}{A_c} d$$

Constant (low frequency) disturbance eliminated if $R(1) = 0$
Closed-loop disturbance response $d \rightarrow u$

$$u = -\frac{SC}{A_c} d$$

Noise (high frequency) disturbance eliminated if $S(-1) = 0$

Pre-chosen factors R_f and S_f

Find solution $AR + BS = A_c$ such that $R = R_1 R_f$ and $S = S_1 S_f$

Polynomial equation with fixed factors

Associate fixed factors to process $A' = AR_f,\ B' = BS_f$

$$A'R_1 + B'S_1 = A_c, \rightarrow \begin{cases} R_1 \\ S_1 \end{cases}$$

with degree conditions

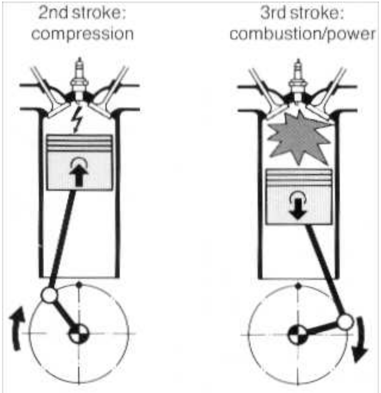
$$\begin{aligned} \deg A_c &\leq \deg A'B' - 1 \\ \deg R_1 &= \deg B' - 1 \\ \deg S_1 &= \deg A' - 1 \end{aligned}$$

Associate fixed factors back to controller

$$\begin{cases} R = R_1 R_f \\ S = S_1 S_f \end{cases}$$

Example: Ignition control of a car engine

For an optimal performance the pressure peak position (PPP) should be kept constant at a given crank angle.



Example: ignition control...

Model

$$y(k) = b_1 u(k-1) + d \quad \begin{cases} k & \text{combustion cycle} \\ y & PPP \\ u & \text{ignition crank angle} \\ d & \text{low frequency disturbance} \end{cases}$$

Controller design with integral action $R_f = 1 - \mathrm{q}^{-1}$
Model $A = 1$, $B = b_1 \mathrm{q}^{-1}$ and $A' = AR_f$

$$\begin{aligned} &\deg A_c \le \deg A' + \deg B - 1 = 1, && A_c = 1 - \lambda \mathrm{q}^{-1} \\ &\deg R_1 = \deg B - 1 = 0, && R_1 = 1, \quad R = R_f = 1 - \mathrm{q}^{-1} \\ &\deg S = \deg A' - 1 = 0, && S = s_0 \end{aligned}$$

Example: ignition control...

Polynomial equation

$$A'R_1 + BS = (1 - \mathrm{q}^{-1}) + b_1 \mathrm{q}^{-1} s_0 = 1 - (1 - b_1 s_0)\mathrm{q}^{-1} = A_c = 1 - \lambda \mathrm{q}^{-1}$$

Design by choosing closed-loop pole λ

$$s_0 = (1 - \lambda)/b_1$$

Choose T for steady-state gain 1

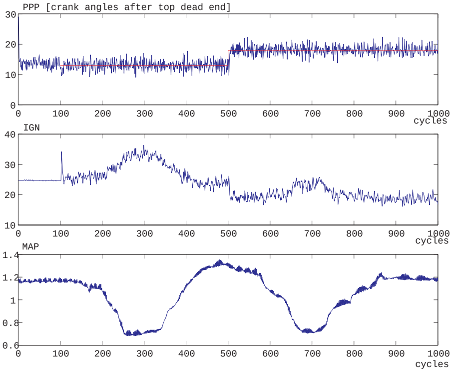
$$\frac{BT}{A_c}(1) = 1 \rightarrow T = \frac{A_c}{B}(1) = \frac{A(1)R(1) + B(1)S(1)}{B(1)} = S(1) = s_0$$

Integral controller ($Ru = -Sy + Tr = s_0 e$)

$$u = \frac{s_0}{1 - \mathrm{q}^{-1}} e$$

Example: experimental results

PPP Pressure peak position, y and setpoint (red)
IGN Ignition before top dead center, $-u$
MAP Manifold air pressure, (low frequency) disturbance d



Example: ignition control—why not PI?

PI-control structure

$$\frac{S}{R} = \frac{s_0 + s_1 q^{-1}}{1 - q^{-1}}$$

Closed-loop characteristic polynomial

$$AR+BS = (1-q^{-1})+b_1 q^{-1}(s_0+s_1 q^{-1}) = 1+(b_1 s_0-1)q^{-1}+b_1 s_1 q^{-2}$$

Compare with

$$A_c = (1-\lambda_1 q^{-1})(1-\lambda_2 q^{-1}) = 1-(\lambda_1+\lambda_2)q^{-1}+\lambda_1\lambda_2 q^{-2}$$

With backward-difference approximation $s_1 = -K$

$$\lambda_1\lambda_2 = b_1 s_1 = -b_1 K < 0 \Rightarrow \lambda_1 > 0, \quad \lambda_2 < 0$$

Why not PI for this problem?

One nasty oscillating pole on negative real axis!

Example: Plate and fan

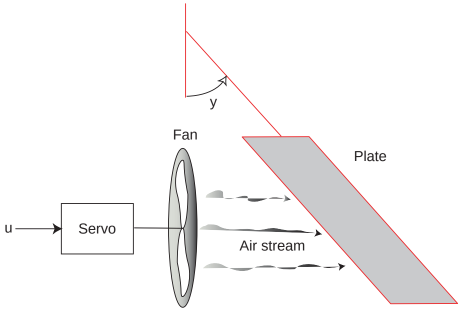
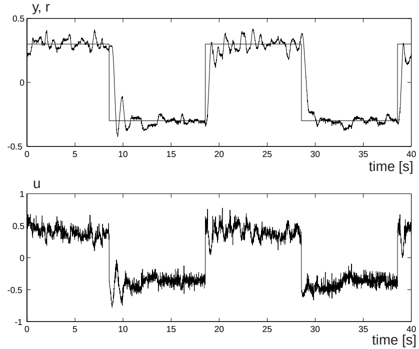


Plate and fan: PID design (Ziegler-Nichols)

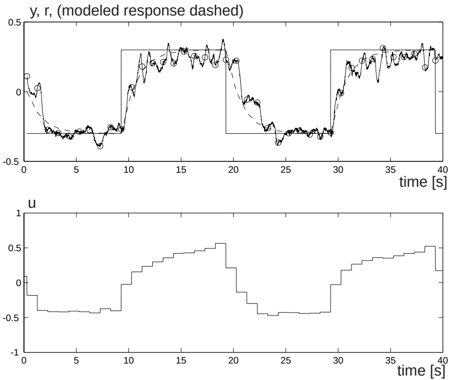


PID (sampling $h = 0.01s$)

Estimate: steady-state gain

$$\frac{\Delta y}{\Delta u} \approx \frac{0.6}{0.8} = b_1$$

Plate and fan: Discrete-time design—tuned slow



Discrete model ($h = 1s$)

$$y(k) = b_1 u(k-1)$$

I-controller

$$u(k) = \frac{s_0}{1 - \mathrm{q}^{-1}} e(k)$$

gives closed-loop response

$$y(k) = \frac{(1-\lambda)\mathrm{q}^{-1}}{1 - \lambda \mathrm{q}^{-1}} r(k)$$

Design $\lambda = 1 - b_1 s_0 = 0.5$

Plate and fan: Discrete-time design—tuned faster

Faster design $\lambda = 0.25$

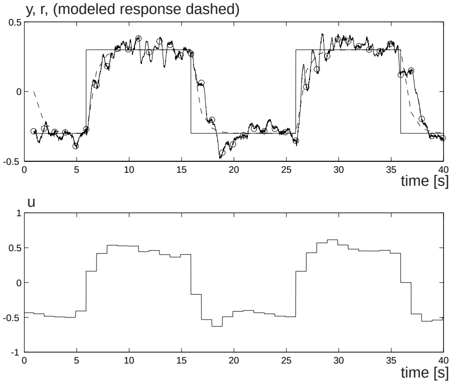
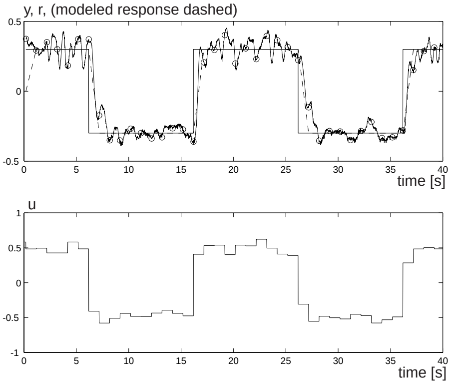


Plate and fan: Discrete-time design—tuned fastest

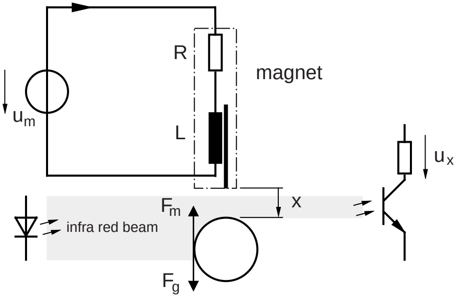


Fastest design (dead-beat)
$\lambda = 0 \Rightarrow y(k) = r(k-1)$

Discrete design compared to discretized continuous design

Same performance but with 100 times slower sampling

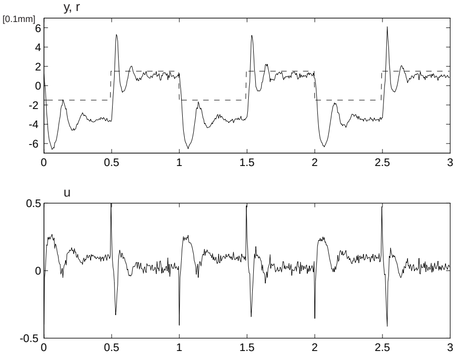
Example: Magnetic suspension system



Dynamics of suspending ball

$$m\ddot{x} = F_g - F_m \qquad \begin{cases} F_g = mg & \text{gravitational force} \\ F_m = -\frac{1}{2}\frac{dL}{dx}i^2 & \text{magnetic force} \end{cases}$$

Example: Magnetic suspension system—PD-control



Example: Magnetic suspension system—Discrete design

Linearized dynamics

$$G(s) = \frac{b}{s^2 - a}$$

Zero-order-hold sampling

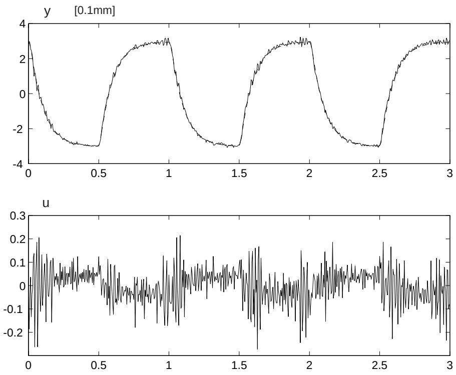
$$\frac{B(\mathrm{q}^{-1})}{A(\mathrm{q}^{-1})} = \gamma \frac{\mathrm{q}^{-1} + \mathrm{q}^{-2}}{1 - \beta \mathrm{q}^{-1} + \mathrm{q}^{-2}}, \qquad \beta > 0$$

Controller design with integral action $R_f = 1 - \mathrm{q}^{-1}$
Controller structure

$$\begin{aligned}
&\deg A_c \le \deg A'B - 1 = 4, && A_c = (1 - \lambda_1 \mathrm{q}^{-1}) \ldots (1 - \lambda_4 \mathrm{q}^{-1}) \\
&\deg R_1 = \deg B - 1 = 1, && R = R_1 R_f = (1 + r_1 \mathrm{q}^{-1})(1 - \mathrm{q}^{-1}) \\
&\deg S = \deg A' - 1 = 2, && S = s_0 + s_1 \mathrm{q}^{-1} + s_2 \mathrm{q}^{-2}
\end{aligned}$$

Pole placement $\lambda_k = 0.95, 0.54, 0.33, 0.21,\ k = 1, \ldots, 4$

Example: Magnetic suspension system—Discrete design



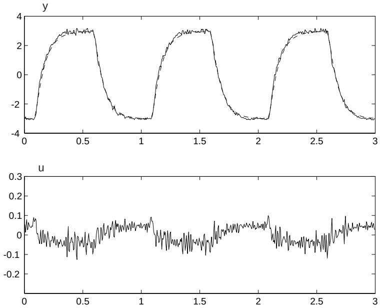
Example: Magnetic suspension system—Discrete design

Reduce noise feedback by choosing $S_f = 1 + \mathrm{q}^{-1}$ $(S_f(-1) = 0)$
Keep integral action $R_f = 1 - \mathrm{q}^{-1}$
Controller structure

$$
\begin{aligned}
\deg A_c &\le \deg A'B' - 1 = 5, & A_c &= (1 - \lambda_1 \mathrm{q}^{-1}) \ldots (1 - \lambda_5 \mathrm{q}^{-1}) \\
\deg R_1 &= \deg B' - 1 = 2, & R_1 &= 1 + r_1 \mathrm{q}^{-1} + r_2 \mathrm{q}^{-2} \\
\deg S_1 &= \deg A' - 1 = 2, & S_1 &= s_0 + s_1 \mathrm{q}^{-1} + s_2 \mathrm{q}^{-2}
\end{aligned}
$$

Higher complexity: $R = R_1 R_f$, $S = S_1 S_f$ both of degree 3

Example: Discrete design with reduced noise feedback



Compare trade-off between tracking and control effort

S_f	$std(e_u)$ $[\mu]$	$\max\lvert\Delta u\rvert$
1	10	0.45
$1 + \mathrm{q}^{-1}$	15	0.12

IMC design

Process with input-disturbance and badly damped poles

$$y(k) = \frac{\mathrm{q}^{-1}}{(1-\lambda_1 \mathrm{q}^{-1})(1-\lambda_2 \mathrm{q}^{-1})}[u(k)+d(k)], \quad \lambda_{1,2} = 0.1 \pm i0.99$$

IMC design

$$\begin{aligned} A_c &= ABq = A \\ R &= Bq(1-\mathrm{q}^{-1}) = (1-\mathrm{q}^{-1}) \\ S &= T = A \end{aligned}$$

Closed-loop response

$$y = \frac{BT}{A_c} r + \frac{BR}{A_c} d = \mathrm{q}^{-1} r + \frac{\mathrm{q}^{-1}(1-\mathrm{q}^{-1})}{A} d$$

Pole-placement design

Polynomial equation

$$AR + BS = A_c$$

Integral action $R_f = 1 - \mathrm{q}^{-1}$ $(A' = AR_f,\ R = R_1 R_f)$

$$\begin{aligned}
&\deg A_c = \deg A'B - 1 = 3 \\
&\deg R_1 = \deg B - 1 = 0, \quad R = R_1 R_f = 1 - \mathrm{q}^{-1} \\
&\deg S = \deg A' - 1 = 2
\end{aligned}$$

Same reference response as IMC if $T = A_c$

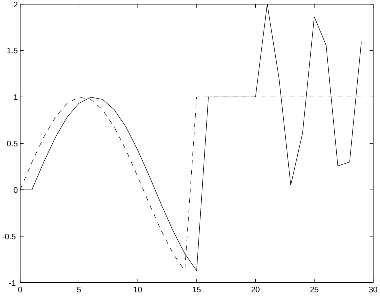
$$y = \frac{BT}{A_c} r + \frac{BR}{A_c} d = \mathrm{q}^{-1} r + \frac{\mathrm{q}^{-1}(1 - \mathrm{q}^{-1})}{A_c} d$$

Pole-placement design versus IMC

Reference r (dashed) and step disturbance d at 20

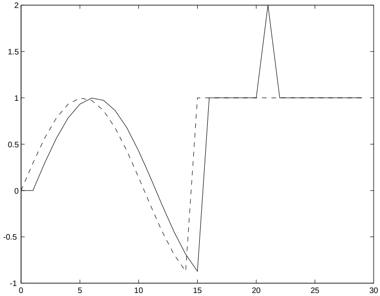
IMC response

$$y = \mathrm{q}^{-1} r + \frac{(1-\mathrm{q}^{-1})}{A} d$$



Pole-placement $A_c = 1$

$$y = \mathrm{q}^{-1} r + \mathrm{q}^{-1}(1-\mathrm{q}^{-1}) d$$



Model matching

Sometimes pole placement design is presented as *model matching*:
Given desired reference model

$$y = Hr, \quad H = \frac{B_H}{A_H}$$

Matching to closed-loop reference response

$$\frac{B_H}{A_H} = \frac{BT}{A_c} \Rightarrow \begin{cases} T = B_H A_o \\ A_c = A_H A_o B \end{cases}$$

Pole placement: $A_H A_o$ chosen freely but not B
Misleading design approach!
Better to choose A_c than B_H

Example

A DC motor with transfer function

$$G(s) = \frac{4}{s(s+2)}$$

is sampled ($h = 0.025s$) to discrete-time model

$$\frac{B}{A} = \frac{1.23 \cdot 10^{-3} \mathrm{q}^{-1}(1 + 0.98\mathrm{q}^{-1})}{(1 - \mathrm{q}^{-1})(1 - 0.95\mathrm{q}^{-1})}$$

Study three design choices:

a) Choose reference model $H = \frac{0.2\mathrm{q}^{-1}}{1 - 0.8\mathrm{q}^{-1}}$
b) Choose pole placement $A_c = 1 - 0.8\mathrm{q}^{-1}$
c) Choose pole placement at 0.95, 0.93 and 0.9

Example: problem a)

Pole placement includes factor $B_2 = 1 + 0.98\mathrm{q}^{-1}$

$$A_c = (1 - 0.8\mathrm{q}^{-1})(1 + 0.98\mathrm{q}^{-1}) \Rightarrow R = R_1 B_2$$

Cancel common factor B_2

$$AR_1 + B_1 S = A_{c1} \quad \begin{cases} \deg R_1 = \deg B_1 - 1 = 0, & R = R_1 B_2 = B_2 \\ \deg S = \deg A - 1 = 1, & S = s_0 + s_1 \mathrm{q}^{-1} \end{cases}$$

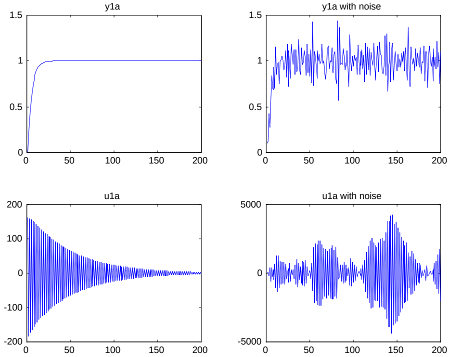
Evaluate at poles ($A(1) = A(0.95^{-1}) = 0$)

$$\left. \begin{array}{l} B(1)S(1) = A_c(1) \\ B(0.95^{-1})S(0.95^{-1}) = A_c(0.95^{-1}) \end{array} \right\} \Rightarrow S = 935 - 772\mathrm{q}^{-1}$$

Adjust steady-state gain $T = A_c(1)/B(1) = 163$

Example: a)

Response without and with measurement noise



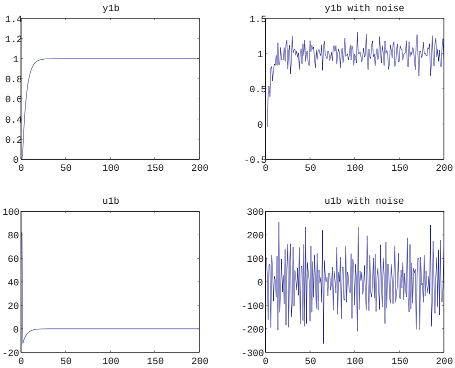
$$A_c = (1 - 0.8\mathrm{q}^{-1})\underbrace{\color{red}{(1 + 0.98\mathrm{q}^{-1})}}_{B_2}$$

$$y = \frac{BT}{A_c} r + \frac{RA}{\color{red}{A_c}} d$$
$$u = \frac{AT}{\color{red}{A_c}} r - \frac{SA}{\color{red}{A_c}} d$$

Pole at -0.98 not canceled in control signal response
$\rightarrow$ Rippling of control signal

Example: b)

Response without and with measurement noise



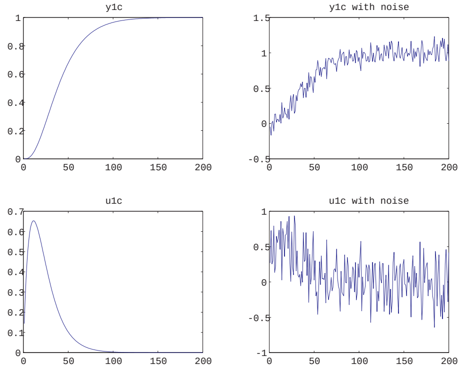
$$A_c = 1 - 0.8q^{-1}$$

$$y = \frac{BT}{A_c} r + \frac{RA}{A_c} d$$
$$u = \frac{AT}{A_c} r - \frac{SA}{A_c} d$$

Well damped pole at 0.8
$\rightarrow$ No rippling of control signal

Example: c)

Response without and with measurement noise



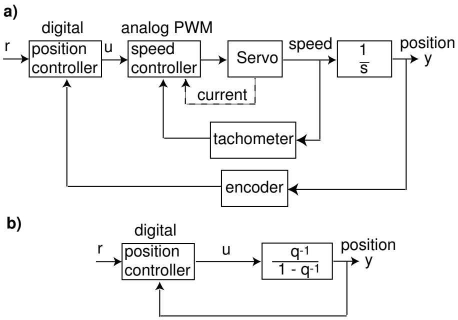
$$A_c(\lambda^{-1}) = 0,$$
$$\lambda = 0.95, 0.93, 0.9$$

$$y = \frac{BT}{A_c} r + \frac{RA}{A_c} d$$
$$u = \frac{AT}{A_c} r - \frac{SA}{A_c} d$$

Noise gain $|\frac{SA}{A_c}(-1)| \approx 3$

Compare **a)** $2 \cdot 10^5$, **b)** 2000

Servo control designs



Discrete model

$$y(k) = y(k-1) + u(k-1), \quad A = 1 - \mathrm{q}^{-1}, B = \mathrm{q}^{-1}$$

Study designs with integral action $R_f = 1 - \mathrm{q}^{-1}$

R and S

Degree conditions ($A' = AR_f$)

$$\begin{aligned}
&\deg A_c = \deg A'B - 1 = 2, && A_c = (1-\lambda_1 \mathrm{q}^{-1})(1-\lambda_2 \mathrm{q}^{-1}) \\
&\deg R_1 = \deg B - 1 = 0, && R = R_1 R_f = (1-\mathrm{q}^{-1}) \\
&\deg S = \deg A' - 1 = 1, && S = s_0 + s_1 \mathrm{q}^{-1}
\end{aligned}$$

Since R known, S is solved directly as

$$\begin{aligned}
A_c &= 1 + a_{c_1}\mathrm{q}^{-1} + a_{c_2}\mathrm{q}^{-2} \\
AR &= (1-\mathrm{q}^{-1})^2 = 1 - 2\mathrm{q}^{-1} + \mathrm{q}^{-2} \\
S &= B^{-1}(A_c - AR) = (a_{c_1} + 2) + (a_{c_2} - 1)\mathrm{q}^{-1}
\end{aligned}$$

Since $a_{c_1} = -\lambda_1 - \lambda_2$ and $a_{c_2} = \lambda_1 \lambda_2$

$$\begin{aligned}
s_0 &= 2 - \lambda_1 - \lambda_2 \\
s_1 &= \lambda_1 \lambda_2 - 1
\end{aligned}$$

T=S (PI structure)

Choosing PI structure $T = S$:

$$u = \frac{S}{R}e = \frac{s_0 + s_1 \mathrm{q}^{-1}}{1 - \mathrm{q}^{-1}}e$$

Reference response with zero at $z_1 = -s_1/s_0$

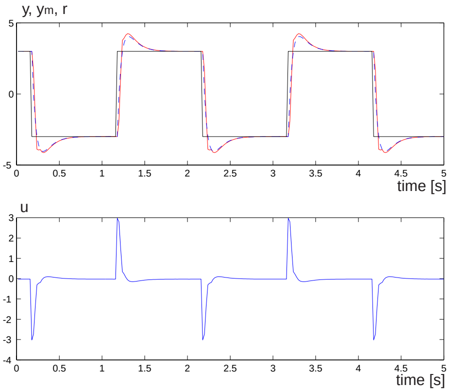
$$y = \frac{BS}{A_c}r = \frac{s_0 + s_1 \mathrm{q}^{-1}}{(1 - \lambda_1 \mathrm{q}^{-1})(1 - \lambda_2 \mathrm{q}^{-1})}\mathrm{q}^{-1}r$$

Pole-placement design

$$\left.\begin{aligned} \lambda_1 &= 0.7 \\ \lambda_2 &= 0.8 \end{aligned}\right\} \Rightarrow z_1 = 0.88$$

The zero causes overshoot!

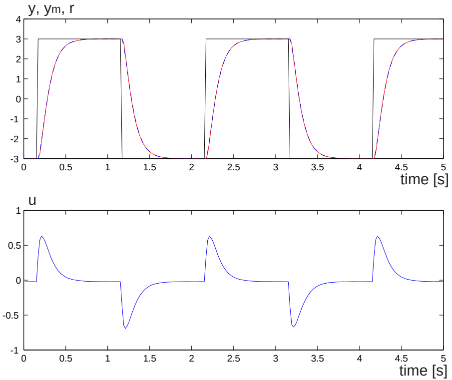
PI structure



The zero causes overshoot!

T scalar

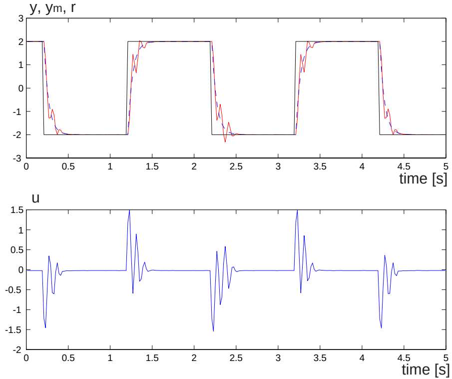
Choose instead $T = S(1) = s_0 + s_1$ scalar $\rightarrow$ no zero introduced!



No overshoot, but slow!

Faster response

Faster response with pole placement $\lambda_1 = 0.7$, $\lambda_2 = 0$



But, unmodeled dynamics excited!

Faster response with less noise feedback

Measurement noise feedback to control signal

$$u = -\frac{AS}{A_c} d$$

Make this more low pass by the slow pole $\lambda_2 = 0.8$

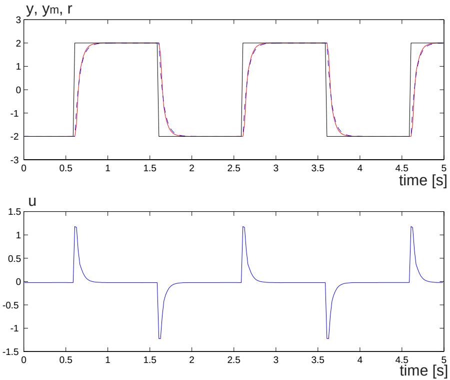
$$A_c = (1 - 0.7\mathrm{q}^{-1})(1 - 0.8\mathrm{q}^{-1})$$

Fast response by canceling the slow pole

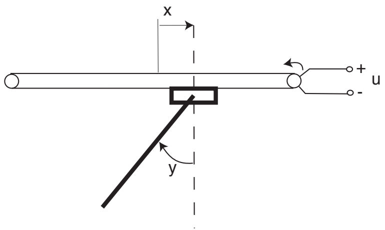
$$T = (1 - 0.8\mathrm{q}^{-1})S(1)/0.2$$

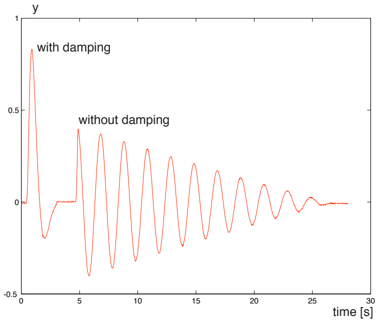
Steady-state gain is one: $T(1) = A_c(1)/B(1) = S(1)$

Faster response with less noise feedback

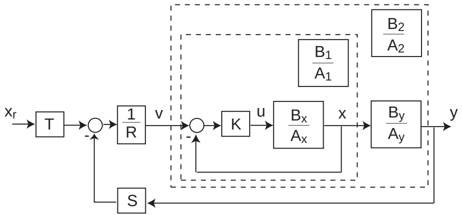


Damping of a pendulum





Cascade control



Trailer dynamics $A_x x = B_x u$

P controller $u = Ke$, $e = v - x$

Closed loop $A_1 x = B_1 v$

$$\begin{cases} A_1 = A_x + KB_x \\ B_1 = KB_x \end{cases}$$

Pendulum dynamics $A_y y = B_y x$

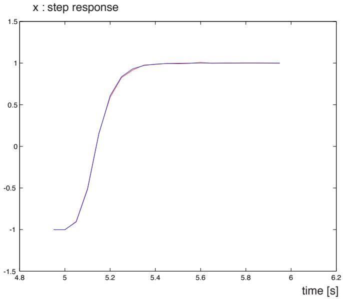
Outer loop model

$$\begin{cases} A_2 = A_1 A_y \\ B_2 = B_1 B_y \end{cases} \quad \begin{cases} A_1 \text{ well damped} \\ A_y \text{ badly damped} \end{cases}$$

Trailer control

Step response for inner loop (modeled and measured response)

$$x = \frac{B_1}{A_1} v$$

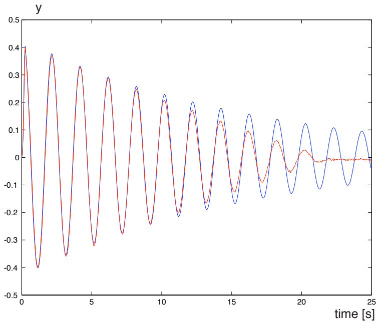


Pendulum model

Pendulum dynamics $A_y y = B_y x$

$$\begin{cases} A_y = (1 - \lambda \mathrm{q}^{-1})(1 - \lambda^* \mathrm{q}^{-1}) \\ B_y = b_0 (1 - \mathrm{q}^{-1})^2 \end{cases}$$

complex poles λ, $\lambda^* = \mathrm{conj}[\lambda]$ close to unit circle



Outer loop feedback design

Keep A_1 (well damped) in pole placement

$$\begin{cases} A_c = A_1 A_{c1} \\ S = A_1 S_1 \end{cases}$$

Polynomial equation (cancel common factor A_1)

$$A_y R + S_1 B_2 = A_{c1}$$

$$\begin{cases} \deg A_{c1} \leq \deg A_y + \deg B_2 - 1 = 6 \\ \deg R = \deg B_2 - 1 = 4 \\ \deg S_1 = \deg A_y - 1 = 1 \end{cases}$$

Chosen well damped poles

$$A_{c1} = \prod_{k=1}^{6} (1 - \lambda_k \mathrm{q}^{-1})$$
$$\lambda_k = 0.85, 0.8, 0.75, 0.7, 0.65, 0.6$$

Reference to trailer position

Reference to trailer position x

$$v(\infty) = \frac{A_2 T}{A_c}(1) r(\infty), \quad \rightarrow T = \frac{A_c}{A_2}(1) = R(1)$$