Exercise 5

Sampling and identification.

Sampling

Open the Sysquake file `sampling.sq` in Sysquake. Two plots will show up, one showing step responses and another showing frequency responses, for the continuous-time system with transfer function

$$G(s) = \frac{202}{s^3 + 4s^2 + 105s + 202}$$

In the same plots, the corresponding discrete-time system responses are displayed. In the *Settings*-menu, it is possible to select how the discrete-time system is calculated. Among the choices there are Zero-order-hold (which exactly calculate the discrete-time system) and various discretizations such as bilinear, backward- and forward-difference approximations. In this problem, the rule of thumb for the selection of sampling frequency is investigated. The many suggestions given in literature can be summarized as

$$\omega_s = \sigma\omega_b, \quad \sigma \in (6, 40)$$

where ω_s is the sampling frequency and ω_b is the bandwidth, defined as the frequency after which the amplitude response remains below 0.7 (if the steady-state gain is 1). The continuous-time frequency response $|G(i\omega)|$ is shown in black. To verify how this is calculated, you may re-plot the same curve by typing in the command window:

```
w = 0:0.01:60;
iw = i*w;
Giw = 202./(iw.^3 + 4*iw.^2 +105*iw + 202);
plot(w,abs(Giw)),
```

Notice, that dot-operations are needed to make element-wise operations. The corresponding discrete-time system $|H(\mathrm{e}^{-\mathrm{i}\omega h})|$ is shown in red. Notice, that it is periodic with period ω_s, since $\omega_s h = 2\pi$ making $\mathrm{e}^{-\mathrm{i}(\omega+\omega_s)h} = \mathrm{e}^{-\mathrm{i}\omega}$. The bandwidth is $\omega_b \approx 10$ [rad/s]. According to the rules of thumb, the selection of sampling frequency should be between 60 and 400 [rad/s]. Select suitable (in your opinion) sampling frequencies for the four cases:

a) zero-order-hold calculation (exact) *[Settings: Zero-order-hold]*

b) bilinear (Tustin) approximation *[Settings: Bilinear]*

c) backward-difference approximation *[Settings: Back Rect]*

d) forward-difference approximation *[Settings: For Rect]*

e) Find out the minimum sampling frequency needed to make the forward-difference discretization stable.

f) Study the stability region (in yellow) for discrete-time poles for the different approximations above. *[Layout: choose 4 plot windows and add Plots: Continuous-time poles and Discrete-time poles]*.

g) Type `help c2dm` in the command window to find out how this sampling function works. Then use it to calculate the zero-order-hold sampling of the system above. Use `filter` to plot a discrete-time step response and verify that you get the same as shown by the sysquake file `sampling.sq`.

h) Use the built-in functions `bodemag` and `dbodemag` to plot the Bode magnitude curves for the continuous-time and discrete-time model, respectively.

Identification

Download the sysquake file `Tank.sq` and open it in sysquake. A tank process is there animated and can be controlled manually or by feedback control. The dynamics of the process can be described by the mass balance equation

$$A\frac{dh}{dt} = q_{in} - q_{out}$$

where A is the tank area, h is the height of water, q_{in} and q_{out} are the inflow and outflow of water, respectively. Energy balance (Bernoulli) gives

$$\rho gh = \frac{\rho v^2}{2} \quad \Rightarrow v = \sqrt{2gh}$$

where v is the outlet water velocity. With outlet area a, the outflow then becomes

$$q_{out} = av = a\sqrt{2gh}$$

The pump flow is proportional to the pump voltage V, according to

$$q_{in} = kV$$

The nonlinear tank dynamics are therefore

$$A\frac{dh}{dt} = kV - a\sqrt{2gh} \qquad \rightarrow \frac{dh}{dt} = -\alpha\sqrt{h} + \beta V = f(h, V)$$

where $\alpha = \frac{a\sqrt{2g}}{A}$ and $\beta = \frac{k}{A}$. For simplicity, the values are chosen $\alpha = \beta = 1$ in the dynamics simulated in `Tank.sq`. Equilibrium ($\dot{h} = 0$) for a constant pump voltage V_0 corresponds to the level $h_0 = V_0^2$. Linearization around the equilibrium gives

$$\frac{d\Delta h}{dt} = f(h, V) \approx \frac{df}{dh}(h_0, V_0)\Delta h + \frac{df}{dV}(h_0, V_0)\Delta V$$

where with notation $y = \Delta h = h - h_0$ and $u = \Delta V = V - V_0$

$$\frac{dy}{dt} = py + du, \qquad \begin{cases} p = -\frac{1}{2\sqrt{h_0}} \\ d = 1 \end{cases}$$

Zero-order-hold sampling with sampling period $h_s = 1$ gives

$$y(k) = \lambda y(k-1) + bu(k-1), \qquad \begin{cases} \lambda = e^{ph_s} = e^p \\ b = (\lambda - 1)/p \end{cases}$$

In polynomial form $A(\mathrm{q}^{-1}) = 1 - \lambda \mathrm{q}^{-1}$ and $B(\mathrm{q}^{-1}) = b\mathrm{q}^{-1}$. The parameters λ and b can be estimated experimentally by the least-squares method. Excite the system around the equilibrium h_0, collect input and output data, solve the least-squares problem as described below. Collect 100 data samples and form the equation system

$$\begin{pmatrix} y(2) \\ \vdots \\ y(100) \end{pmatrix} = \begin{pmatrix} y(1) & u(1) \\ \vdots & \vdots \\ y(99) & u(99) \end{pmatrix} \begin{pmatrix} \lambda \\ b \end{pmatrix} + \begin{pmatrix} e(2) \\ \vdots \\ e(100) \end{pmatrix}$$

or in matrix form $\mathbf{y} = W\theta + \mathbf{e}$, where $\mathbf{e}$ is a vector of equation errors. Minimization of the squared equation error $\sum e(k)^2 = \mathbf{e}^\mathbf{T}\mathbf{e}$ results in the analytical solution

$$\theta = (W^T W)^{-1} W^T y$$

Now do the following experiment with the tank process. To access the samples, first write in the command window

```
> global samples
```

The variable `samples` will now continuously change with the data from the simulation. It contains 100 of the latest data samples, as `samples=[t, u, y]`, where t is the time (in seconds), u the input samples and y the output samples.

a) Excite the tank system manually using u_0 ($K = 0$) such that y varies with mean close to $h_0 = 10$. After 100 seconds (samples) store data into a variable and form the equation system:

```
> s=samples;
> y=s(2:100,3); y1=s(1:99,3); u=s(2:100,2); u1=s(1:99,2);
> y=y-mean(y); y1=y1-mean(y1); u=u-mean(u); u1=u1-mean(u1);
> W = [y1 u1]; th = (W'*W)\W'*y
```

Calculate $\theta = [\lambda, \quad b]^T$ from sampling of the system. Is the estimated θ close to this theoretical one?

b) Investigate how good the estimated model is by plotting the real output y together with the prediction output based on the model, i.e. $\hat{y} = W\theta$.

```
> yh=W*th;
> plot(yh','b');  plot(y')
```

c) A better evaluation is to compare with the output from a simulation of the system.

```
> B=[0 th(2)]; A=[1 -th(1)];
> ys=filter(B,A,u); plot(ys','r'); plot(y')
```

d) Repeat the above experiment with excitation around $h_0 = 350$. Calculate the theoretical θ and compare it to what you can estimate experimentally.