Seminar 1: Difference equations

January 15, 2007

Outline

Discrete-time control

Digital system

Discrete-time $u(k), y(k)$, $k = 0, 1, 2, \dots$

Discrete values a signal can take (finite precision)

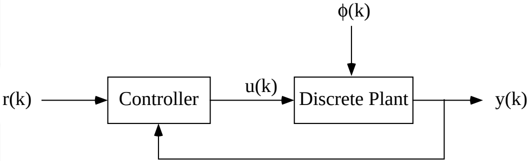
Example

Car engine control

$u(k)$ = fuel injection time

$y(k)$ = air/fuel ratio

k = combustion cycle

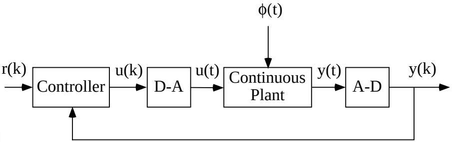


Discrete-time control of continuous-time systems

- analogue signals $u(t)$, $y(t)$
- real time $t \in \Re$
- discrete-time signals $u(k)$, $y(k)$
- discrete time $k = 0, 1, 2, \ldots$

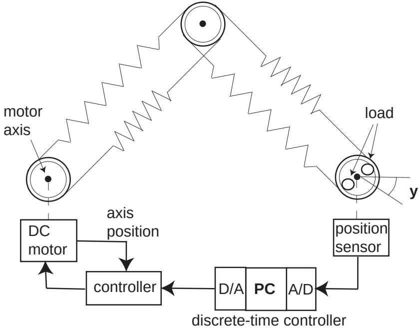
Example

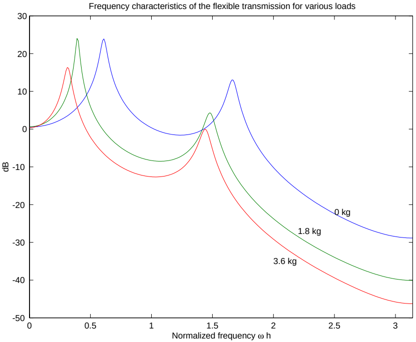
Tank level control
$u(k)$ = pump voltage
$y(k)$ = tank level
k = index of sampling instant



Discrete-time system includes D-A and A-D converters

A challenging benchmark example





You will learn how to make a model-based controller design handling with changing dynamics

Linear difference equations

A linear difference equation describing a signal y

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n)$$

With input u and output y

$$y(k) = -a_1 y(k-1) - \ldots a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \ldots$$

With two inputs, u_1 and u_2, and one output y

$$y(k) = -a_1 y(k-1) \ldots b_0 u_1(k) + b_1 u_1(k-1) + \ldots c_0 u_2(k) + c_1 u_2(k-1) + \ldots$$

where $a_1, a_2, \ldots, b_0, b_1 \ldots, c_0, c_1 \ldots$ are real constants

Example

$$y(k) = -0.9y(k-1) + 0.1u(k-1)$$

Initial conditions $y(0) = u(0) = 0$
Response of the system for the input $u(1) = 1$, $u(2) = -1$

$$y(1) = -0.9y(0) + 0.1u(0) = -0.9 \cdot 0 + 0.1 \cdot 0 = 0$$
$$y(2) = -0.9y(1) + 0.1u(1) = -0.9 \cdot 0 + 0.1 \cdot 1 = 0.1$$
$$y(3) = -0.9y(2) + 0.1u(2) = -0.9 \cdot 0.1 + 0.1 \cdot (-1) = -0.19$$

Shift operators

Backward-shift operator

$q^{-1}y(k) = y(k-1)$

implementable (memory)

Forward-shift operator

$qy(k) = y(k+1)$

not implementable (future)

Example

$$
\begin{aligned}
y(k) &= -0.9y(k-1) + 0.1u(k-1) \\
y(k) &= -0.9q^{-1}y(k) + 0.1q^{-1}u(k) \\
(1+0.9q^{-1})y(k) &= 0.1q^{-1}u(k) \\
A(q^{-1})y(k) &= B(q^{-1})u(k)
\end{aligned}
$$

Example

$$
\begin{aligned}
y(k) &= -0.9y(k-1) + 0.1u(k-1) \\
y(k+1) &= -0.9y(k) + 0.1u(k) \\
(q+0.9)y(k) &= 0.1u(k) \\
\bar{A}(q)y(k) &= \bar{B}(q)u(k)
\end{aligned}
$$

Polynomial description

A difference equation in **polynomial form**

$$A(\mathrm{q}^{-1})y(k) = B(\mathrm{q}^{-1})u(k), \quad \begin{cases} A(\mathrm{q}^{-1}) = 1 + a_1\mathrm{q}^{-1} + a_2\mathrm{q}^{-2} + \dots \\ B(\mathrm{q}^{-1}) = b_0 + b_1\mathrm{q}^{-1} + b_2\mathrm{q}^{-2} + \dots \end{cases}$$

or, for brevity,

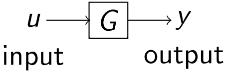
$$Ay = Bu$$

useful for *analysis*.
For *calculation* of responses, use **recursive form**

$$y(k) = -a_1 y(k-1) \dots + b_0 u(k) + b_1 u(k-1) \dots$$

Transfer operator

Input-output description



$$y(k) = G(\mathrm{q}^{-1})u(k), \quad G(\mathrm{q}^{-1}) = \frac{B(\mathrm{q}^{-1})}{A(\mathrm{q}^{-1})}$$

Transfer operator G interpreted as difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \ldots + b_0 u(k) + b_1 u(k-1) \ldots$$

Poles and zeros

$$y = G(\mathrm{q}^{-1})u = \bar{G}(\mathrm{q})u$$

Forward-shift description

zero

z is a zero if $\bar{G}(z) = 0$

pole

λ is a pole if $\bar{G}(\lambda) \to \infty$

Example

$$y(k) = 0.9y(k-1) + u(k) + u(k-1)$$

$$\bar{G}(\mathrm{q}) = \frac{\mathrm{q}+1}{\mathrm{q}-0.9} \Rightarrow \begin{cases} \lambda_1 = 0.9 \\ z_1 = -1 \end{cases}$$

Example

Backward-shift description

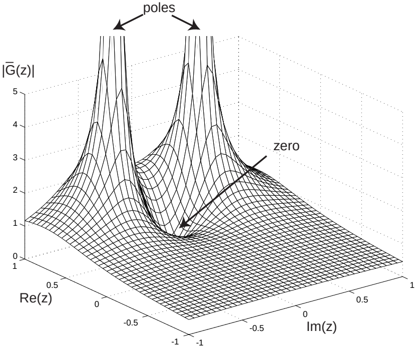
$$A(\mathrm{q}^{-1})y(k) = B(\mathrm{q}^{-1})u(k)$$

with $\lambda_{1,2} = 0.7 \pm 0.4i$, $z_1 = 0.5$

$$\begin{cases} A(\mathrm{q}^{-1}) = (1 - \lambda_1 \mathrm{q}^{-1})(1 - \lambda_2 \mathrm{q}^{-1}) \\ B(\mathrm{q}^{-1}) = \mathrm{q}^{-1}(1 - z_1 \mathrm{q}^{-1}) \end{cases}$$

Forward-shift description

$$\bar{G} = \frac{\mathrm{q} - z_1}{(\mathrm{q} - \lambda_1)(\mathrm{q} - \lambda_2)}$$



Stability definition

$$y(k) + a_1 y(k-1) + \dots a_n y(k-n) = 0$$

The system is stable if

$$y(k) \to 0, k \to \infty$$

for all initial conditions

Example

If $A(\mathrm{q}^{-1}) = 1 - \lambda \mathrm{q}^{-1}$, the general solution is $y(k) = y(0)\lambda^k$

$$A(\mathrm{q}^{-1}) y(k) = y(k) - \lambda y(k-1) = y(0)\lambda^k - \lambda y(0)\lambda^{k-1} = 0$$

Stable if and only if $|\lambda| < 1$

Second order system

$$A(\mathrm{q}^{-1})y(k) = y(k) + a_1 y(k-1) + a_2 y(k-2) = 0$$

Factorize in poles

$$A(\mathrm{q}^{-1}) = 1 + a_1\mathrm{q}^{-1} + a_2\mathrm{q}^{-2} = (1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1})$$

General solution

$$y(k) = c_1\lambda_1^k + c_2\lambda_2^k$$

since $A_i = 1 - \lambda_i\mathrm{q}^{-1}$ and $y_i(k) = c_i\lambda_i^k$ satisfies $A_i y_i = 0$

$$Ay = A_1A_2(y_1+y_2) = A_2\underbrace{[A_1y_1]}_{=0} + A_1\underbrace{[A_2y_2]}_{=0} = 0$$

Stable if and only if $|\lambda_i| < 1,\ i = 1,2$

Example

$$y(k)-2y(k-1)+0.99y(k-2)=0$$

Initial conditions $y(0)=0.9$ and

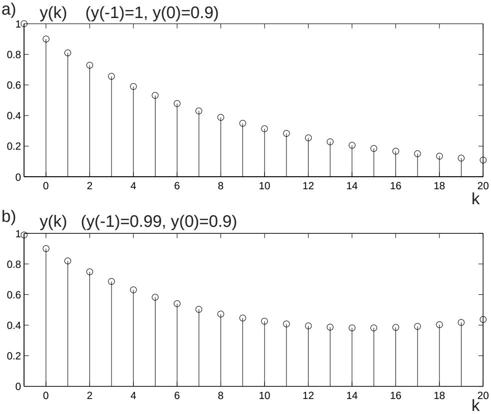
a) $y(-1)=1$

b) $y(-1)=0.99$

Poles: $\lambda_1=0.9,\ \lambda_2=1.1$

$$y(k)=c_1\lambda_1^k+c_2\lambda_2^k$$

In **a)** $c_2=0$, but not in **b)**



Thus, not stable!

General case

Difference equation

$$A(\mathrm{q}^{-1})y(k) = 0$$

factorized in poles

$$A(\mathrm{q}^{-1}) = (1 - \lambda_1 \mathrm{q}^{-1})(1 - \lambda_2 \mathrm{q}^{-1}) \ldots (1 - \lambda_n \mathrm{q}^{-1})$$

Stability criterion

$$|\lambda_i| < 1, \quad i = 1, 2, \ldots, n$$

Steady-state (stationary) gain

Unit step

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Step response

$$y(k) + a_1 y(k-1) + \ldots a_n y(k-n) = b_0 \underbrace{u(k)}_{=1} + b_1 \underbrace{u(k-1)}_{=1} +$$

If stable $y(k) \to y_\infty$, $k \to \infty$

$$y_\infty = \frac{b_0 + b_1 + \ldots}{1 + a_1 + a_2 + \ldots} = \frac{B(1)}{A(1)} = G(1)$$

Steady-state gain

If stable, then $\frac{y_\infty}{u_\infty} = G(1)$

Example: FIR

Unit pulse (impulse)

$$u(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Pulse response for $y(k) = 5u(k) + 2u(k-1)$

$$\begin{aligned} y(0) &= 5u(0) + 2u(-1) = 5 \cdot 1 + 2 \cdot 0 = 5 \\ y(1) &= 5u(1) + 2u(0) = 5 \cdot 0 + 2 \cdot 1 = 2 \\ y(2) &= 5u(2) + 2u(1) = 5 \cdot 0 + 2 \cdot 0 = 0 \\ y(k) &= 0, \quad k > 1 \end{aligned}$$

Finite Impulse Response (FIR)
Pulse response for $y(k) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m)$

$$y(k) = b_k$$

Example: Step response

Example

Step response for $y(k) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m)$

$$
\begin{aligned}
&y(0) = b_0 \\
&y(1) = b_0 + b_1 \\
&\vdots \\
&y(m) = b_0 + b_1 + \ldots + b_m = B(1) \\
&y(k) = B(1) \quad k \geq m
\end{aligned}
$$

Example: First order system

Example

Step response for $y(k) = 0.9y(k-1) + 0.1u(k)$

$$
\begin{aligned}
y(0) &= 0.1 \\
y(1) &= 0.9 \cdot 0.1 + 0.1 = 0.19 \\
y(2) &= 0.9 \cdot 0.19 + 0.1 = 0.271 \\
&\vdots \\
y(k) &\to \frac{B(1)}{A(1)} = 1, \quad k \to \infty
\end{aligned}
$$

Compare analytical solution

$$
\begin{aligned}
y(k) &= c_1 \lambda_1^k + 1 \\
y(0) &= c_1 0.9^0 + 1 = 0.1 \to c_1 = -0.9 \\
\to y(k) &= -0.9 \cdot 0.9^k + 1
\end{aligned}
$$

Example: First order system

Step response for

$$y(k) = -0.9y(k-1) + 1.9u(k)$$

Recursively

$$\begin{aligned}
y(0) &= 1.9 \\
y(1) &= -0.9 \cdot 1.9 + 1.9 = 0.19 \\
y(2) &= -0.9 \cdot 0.19 + 1.9 = 1.73 \\
&\vdots \\
y(k) &\to \frac{B(1)}{A(1)} = 1
\end{aligned}$$

Analytically

$$y(k) = c_1 \lambda_1^k + 1 = 0.9(-0.9)^k + 1$$

Oscillating response when $\lambda_1 < 0$

Example: Second order system

Pulse response for

$$A(\mathrm{q}^{-1})y(k) = B(\mathrm{q}^{-1})u(k)$$

$$A(\mathrm{q}^{-1}) = (1 - \lambda_1 \mathrm{q}^{-1})(1 - \lambda_2 \mathrm{q}^{-1})$$

$$k > \deg B \Rightarrow Bu(k) = 0 \text{ (FIR)}$$

$$y(k) = c_1 \lambda_1^k + c_2 \lambda_2^k$$

For complex poles

$$\begin{aligned} \lambda_1 &= \lambda = |\lambda| e^{i \arg(\lambda)} \\ \lambda_2 &= \lambda^* = |\lambda| e^{-i \arg(\lambda)} \\ c_1 &= c_2^* = |c| e^{i \arg(c)} \end{aligned}$$

$$y(k) = 2|c||\lambda|^k \cos(\arg(\lambda)k + \arg(c))$$

Oscillating response

amplitude: scales by $|\lambda|^k$

frequency: $\omega = \arg(\lambda)$

Simulation of a difference equation

Implement difference equation

$$y(k) = -a_1 y(k-1) + b_1 u(k-1)$$

Edit the text file:

test.sq

```
functions
{@
function y = mydiff(u)
   global y1 u1 a1 b1
   if  isempty(y1),
         y1=0;
         u1=0;
         a1=-0.9;
         b1=0.1;
   end
   y = -a1*y1 + b1*u1;
   y1 = y;
   u1 = u;
@}
```

and open it in Sysquake

Test in **command window**

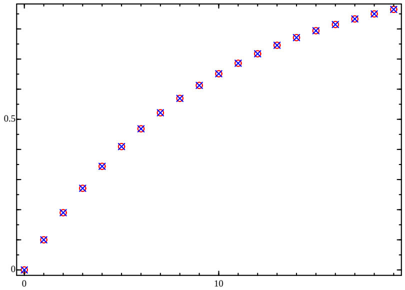
```
> y=mydiff(0)
y =
      0
> y=mydiff(1)
y =
      0
> y=mydiff(-1)
y =
    0.1000
> y=mydiff(0)
y =
   -0.0100
```

Example: compare with built-in function

Test in **command window**

```
> global y1 u1 a1 b1
> y1=0;u1=0;a1=-0.9;b1=0.1;
> k=0:19;
> for n=1:20, y(n)=mydiff(1); end;
> plot(k,y,'ro')
> A=[1 a1]; B=[0 b1];
> yf=filter(B,A,ones(1,20));
> plot(k,yf,'bx')
```

Result in **plot window**

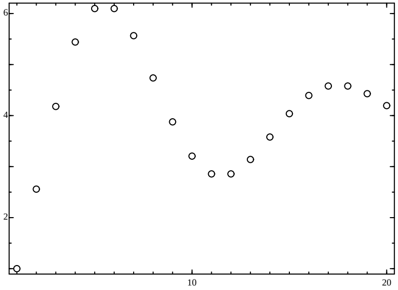


Example: step response of second order system

Test in **command window**

```
> lambda1=0.9*exp(i*pi/6)
lambda1 =
  0.7794 +   0.45j
> lambda2=conj(lambda1)
lambda2 =
  0.7794 -   0.45j
> A=poly([lambda1,lambda2])
A =
        1   -1.5588      0.81
> conv([1 -lambda1],[1 -lambda2])
ans =
      1          -1.5588              0.81
> y=filter(1,A,ones(1,20));
> clf
> plot(y,'o')
> 1/sum(A) % steady-state gain
ans =
    3.9816
```

Result in **plot window**



Notice:

$$\arg \lambda_1 = \frac{\pi}{6} = \omega = \frac{2\pi}{T}$$

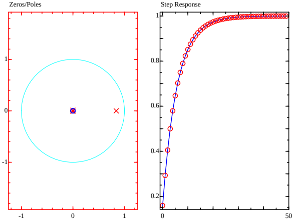
Period $T = 12$ samples

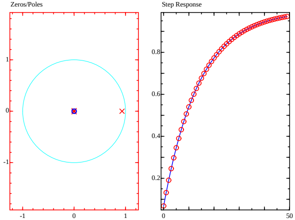
Example: real positive poles

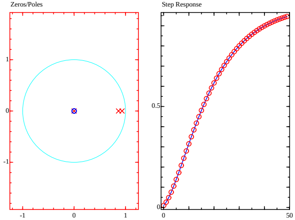
Open **zeropole.sq** and drag poles

$$G(q^{-1}) = \frac{(1 - z_1 q^{-1})(1 - z_2 q^{-1})(1 - z_3 q^{-1})}{(1 - \lambda_1 q^{-1})(1 - \lambda_2 q^{-1})(1 - \lambda_3 q^{-1})}$$

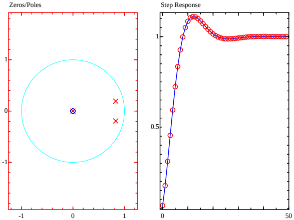
Move λ_1, then λ_2 (other 0)

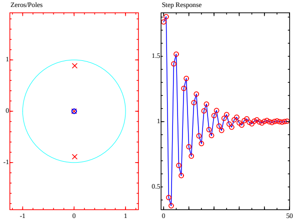


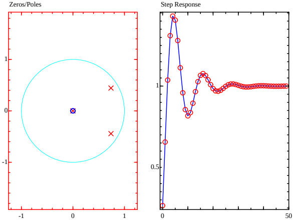


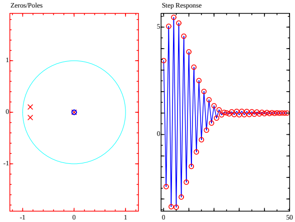


Example: complex poles

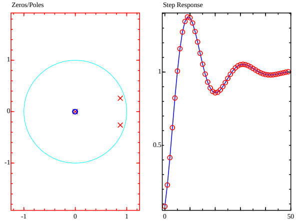


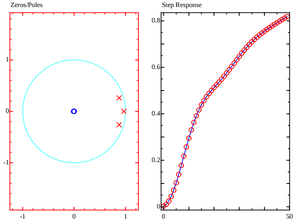






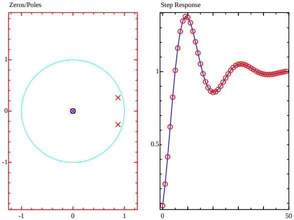
Example: dominant poles

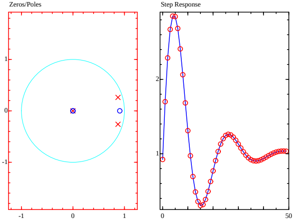


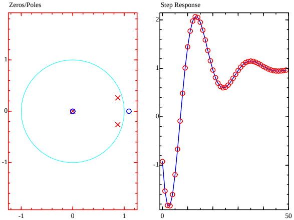


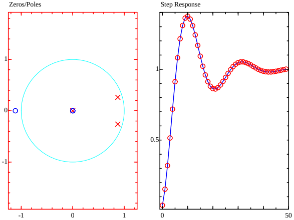
Pole closest to 1 *dominates* step response

Example: influence of zero









Example: influence of zero

Compare step responses (u step)

$$y_1 = G_1 u$$
$$y_2 = G_2 u = G_1 B u$$

$B = b_0 + b_1 \mathrm{q}^{-1} = \frac{1 - z\mathrm{q}^{-1}}{1-z}$
Study, first step response

$$u_2 = Bu \rightarrow \begin{cases} b_0 = \frac{1}{1-z} & k = 0 \\ b_0 + b_1 = 1 & k \geq 1 \end{cases}$$

Now, compare u and u_2 on G_1

$$y_1 = G_1 u$$
$$y_2 = G_1 u_2$$

Scenarios for step responses:
$z = 0.8 \Rightarrow u_2(0) = 5$. Large kick, expected faster and overshoot
$z = -1 \Rightarrow u_2(0) = 0.5$. Smooth step, expected slightly slower
$z = 1.1 \Rightarrow u_2(0) = -10$. Large down-kick, expected undershoot
Conclusion:
$\rightarrow$ zero close to 1 large transient