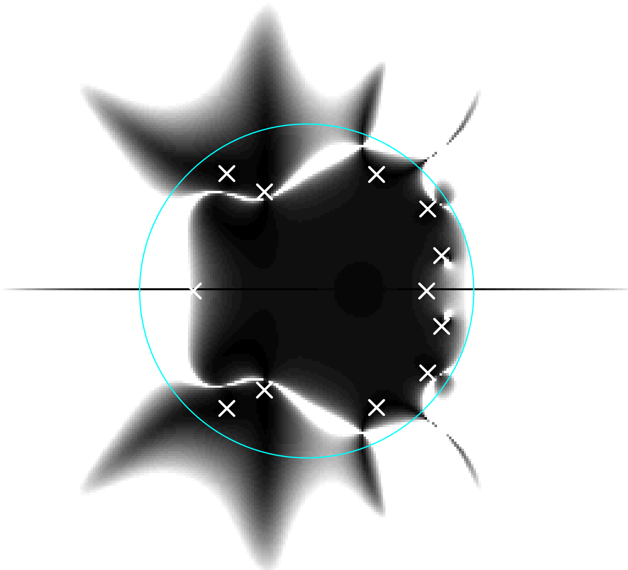
# Digital Control



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# Contents

# Chapter 1

# Introduction

*Control, control, you must learn control*
Yoda Master

## 1.1 Discrete-time Control

Consider the discrete-time control system in Fig. 1.1. Here, the plant as well

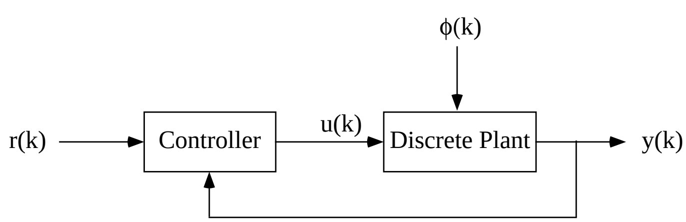



Figure 1.1: A discrete-time control system.

as the controller are discrete-time systems, i.e. the evolution of time is simply counted by an integer $k$. From the controllers viewpoint a continuous-time plant extended with digital-to-analog (D-A) and analog-to-digital (A-D) converters, as in Fig. 1.2, is also a discrete-time system. A model-based controller design can therefore use a discrete-time representation of the plant even though the physical plant is continuous. Loosely speaking, the objective of the control system is to make the output $y$ follow the reference $r$ despite the presence of the disturbance $\varphi$.

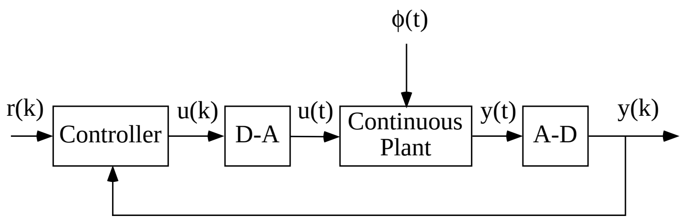



Figure 1.2: Discrete-time control of a continuous-time plant.

## 1.2 Difference Equations and Transfer Operators

A *difference equation* is a recursive relation between an 'output' signal, say $y(k)$, and previous values of it, i.e. $y(k-1), y(k-2), \ldots$, and possibly other external 'input' signals, say $u(k), u(k-1), \ldots$. Thus, it looks like

$$y(k) = f(y(k-1), y(k-2), \ldots, u(k), u(k-1), \ldots)$$

The function $f$ describes how $y$ is incremented. Only *linear* difference equations will be considered here. This means that $f$ only consists of simple operations like multiplications by parameters and additions. A simple example is given below.

**Example: 1.1**

Consider the linear difference equation below where the parameters $a_1$ and $b_1$ are multiplied to the old signal values whereafter these are added together to form the next output.

$$y(k) = -a_1 y(k-1) + b_1 u(k-1)$$

Once the input sequence is given, it is straightforward to calculate the output. Let, for example, the parameters be $a_1 = 0.9$ and $b_1 = 0.1$, and the input sequence be $u(1) = 1$, $u(2) = -1$. Assuming the initial condition to be zero $y(0) = u(0) = 0$, if follows successively that

$$\begin{aligned}
y(1) &= -0.9y(0) + 0.1u(0) = -0.9 \cdot 0 + 0.1 \cdot 0 = 0 \\
y(2) &= -0.9y(1) + 0.1u(1) = -0.9 \cdot 0 + 0.1 \cdot 1 = 0.1 \\
y(3) &= -0.9y(2) + 0.1u(2) = -0.9 \cdot 0.1 + 0.1 \cdot (-1) = -0.19
\end{aligned}$$

To be able to continue calculate $y(k)$, for $k > 3$, $u(k)$, for $k > 2$ are needed. □

In a plant model description two inputs will be considered. One is called control signal $u$ and is to our disposal. The other is called disturbance signal $\varphi$

and can not be directly manipulated. It will be used to describe how disturbances enter and effect the system. Collecting all outputs $y$, inputs $u$ and disturbances $\varphi$ under separate summations the plant model is

$$\sum_{i=0}^{N_A} a_i y(k-i) = \sum_{i=d}^{N_B} b_i u(k-i) + \sum_{i=0}^{N_C} c_i \varphi(k-i) \tag{1.1}$$

There is always a *time delay* $d \geq 1$ for a discrete-time system. Thus, the output, $y(k)$, does not depend on the input at the same instant, $u(k)$. All coefficients are supposed to be fixed. Using the backward-shift operator $\mathrm{q}^{-1}$, defined by $\mathrm{q}^{-1}y(k) = y(k-1)$, the equation becomes:

$$\mathsf{A}(\mathrm{q}^{-1})y(k) = \mathsf{B}(\mathrm{q}^{-1})u(k) + \mathsf{C}(\mathrm{q}^{-1})\varphi(k)$$

with the polynomials defined as

$$\begin{aligned}
\mathsf{A}(\mathrm{q}^{-1}) &= 1 + a_1\mathrm{q}^{-1} + \ldots + a_{\deg \mathsf{A}}\mathrm{q}^{-\deg \mathsf{A}} \\
\mathsf{B}(\mathrm{q}^{-1}) &= b_1\mathrm{q}^{-1} + \ldots + b_{\deg \mathsf{B}}\mathrm{q}^{-\deg \mathsf{B}} \\
\mathsf{C}(\mathrm{q}^{-1}) &= c_0 + c_1\mathrm{q}^{-1} + \ldots + c_{\deg \mathsf{C}}\mathrm{q}^{-\deg \mathsf{C}}
\end{aligned}$$

In order to easier manipulate with algebraic expressions and for transparent display in block schemes it is sometimes useful to reduce the three objects $\mathsf{A}, \mathsf{B}$ and $\mathsf{C}$ into two by dividing with $\mathsf{A}$. We then get an input-output relation

$$y(k) = \mathsf{G}_u(\mathrm{q}^{-1})u(k) + \mathsf{G}_\varphi(\mathrm{q}^{-1})\varphi(k)$$

described by the *transfer operators* (or pulse-transfer operators):

$$\mathsf{G}_u(\mathrm{q}^{-1}) = \frac{\mathsf{B}(\mathrm{q}^{-1})}{\mathsf{A}(\mathrm{q}^{-1})}, \qquad \mathsf{G}_\varphi(\mathrm{q}^{-1}) = \frac{\mathsf{C}(\mathrm{q}^{-1})}{\mathsf{A}(\mathrm{q}^{-1})}$$

This representation can lead to a misinterpretation. For example, $u(k) \equiv \varphi(k) \equiv 0$, $t \geq 0$, gives the impression that $y(k) \equiv 0$, which is not true if the initial conditions $y(0), y(-1), \ldots$ are nonzero. To avoid this problem, the transfer-operator description should be interpreted as just a compact way of writing (1.1). The transfer operators are said to be proper (causal) when the output at time $k$ does not depend on the input at time $k+1, k+2, \ldots$. Here, this is guaranteed by the fact that $A(0) \neq 0$.

Similarly, the controller is described by a difference equation with two inputs: one is the reference signal $r$ which is an external input, and the other is the plant output $y$ which is fed back to the controller. Consequently, such a controller structure is called a *feedback* controller. In polynomial form this is

$$R(\mathrm{q}^{-1})u(k) = T(\mathrm{q}^{-1})r(k) - S(\mathrm{q}^{-1})y(k)$$

with

$$\begin{aligned} R(\mathrm{q}^{-1}) &= 1 + r_1\mathrm{q}^{-1} + \ldots + r_{\deg R}\mathrm{q}^{-\deg R} \\ S(\mathrm{q}^{-1}) &= s_0 + s_1\mathrm{q}^{-1} + \ldots + s_{\deg S}\mathrm{q}^{-\deg S} \\ T(\mathrm{q}^{-1}) &= t_0 + t_1\mathrm{q}^{-1} + \ldots + t_{\deg T}\mathrm{q}^{-\deg T} \end{aligned}$$

The controller is causal since $R(0) \neq 0$. With transfer-operators it is

$$u(k) = G_r(\mathrm{q}^{-1})r(k) - G_y(\mathrm{q}^{-1})y(k)$$

with

$$G_r(\mathrm{q}^{-1}) = \frac{T(\mathrm{q}^{-1})}{R(\mathrm{q}^{-1})}, \qquad G_y(\mathrm{q}^{-1}) = \frac{S(\mathrm{q}^{-1})}{R(\mathrm{q}^{-1})}$$

The system under control, the *closed-loop* (see Fig. 1.3), is described by:

$$y(k) = H_r(\mathrm{q}^{-1})r(k) + H_\varphi(\mathrm{q}^{-1})\varphi(k)$$

with the closed-loop transfer operators

$$\begin{aligned} H_r(\mathrm{q}^{-1}) &= \frac{BT}{AR + BS}, \\ H_\varphi(\mathrm{q}^{-1}) &= \frac{RC}{AR + BS} \end{aligned}$$

where the dependence on $\mathrm{q}^{-1}$ is suppressed for brevity.

The polynomial appearing in the denominator,

$$A_c = AR + BS,$$

is called the *characteristic polynomial*, and its roots can be moved in the complex plane by the controller polynomials $R(\mathrm{q}^{-1})$ and $S(\mathrm{q}^{-1})$.

## 1.3 Poles and zeros

Traditionally, poles and zeros of a discrete-time transfer function are defined from the forward-shift representation. This is convenient since the poles, defined this

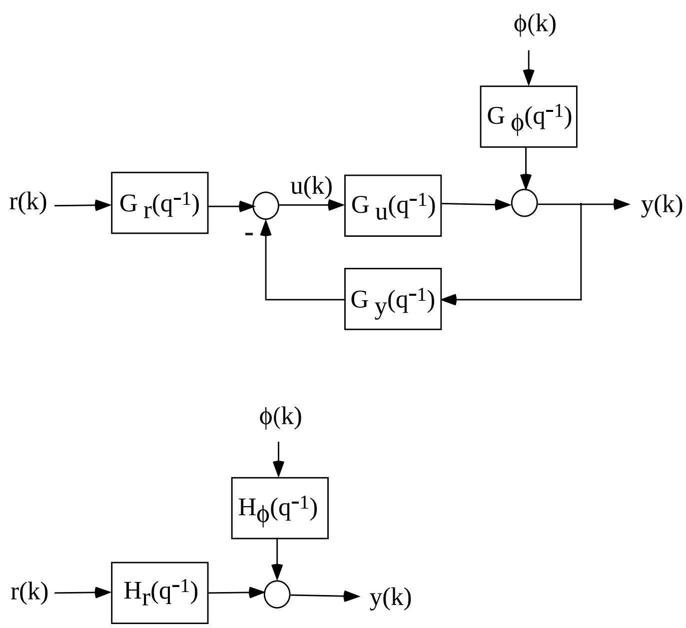



Figure 1.3: A general feedback system.

way, are bounded for stable systems and therefore easier to display graphically. We therefore keep to this tradition. Consider an arbitrary transfer operator

$$G(\mathrm{q}^{-1}) = \frac{\prod_{i=1}^{m} \mathrm{q}^{-\mathrm{d}}(1 - z_i \mathrm{q}^{-1})}{\prod_{i=1}^{n}(1 - \lambda_i \mathrm{q}^{-1})} = \frac{\prod_{i=1}^{m}(\mathrm{q} - z_i)}{\prod_{i=1}^{n}(\mathrm{q} - \lambda_i)} = \bar{G}(\mathrm{q})$$

where the delay is $d = n - m$ and where $\bar{G}(\mathrm{q})$ is the transfer operator expressed in the forward-shift operator $\mathrm{q}$ ($\mathrm{q}y(k) = y(k+1)$. Replace the operator $\mathrm{q}$ by a complex variable $z$. Then $\bar{G}(z)$ can be considered as a complex-valued function—a transfer function. The roots $z_i$, $i = 1, \ldots, n$ of the numerator polynomial of a transfer function are called the *zeros* of the transfer function (or loosely speaking the zeros of the plant, the zeros of the controller or zeros of whatever the transfer function describes). The roots $\lambda_i$, $i = 1, \ldots, n$ of the denominator of a transfer function are called the *poles* (or eigenvalues) of the transfer function (or poles of the object it describes).

**zero:** Evaluation of a transfer function at a zero results in zero: $\bar{G}(z_i) = 0$.

**pole:** Evaluation of a transfer function at a pole results in a 'pole' sticking up to infinity: $\bar{G}(\lambda_i) = \infty$.

The closed-loop poles are thus defined as $\{\lambda : A_c(\lambda^{-1}) = 0\}$. These can be modified by feedback. However, the open-loop zeros remain after feedback, i.e. the polynomials $B(\mathrm{q}^{-1})$ and $C(\mathrm{q}^{-1})$ reappear in the numerators of $H_r(\mathrm{q}^{-1})$ and $H_\varphi(\mathrm{q}^{-1})$, respectively. Zeros can only be removed from the closed-loop transfer function if they are being canceled by corresponding closed-loop poles. Although pole-zero cancellation is sometimes possible it is usually a bad strategy.

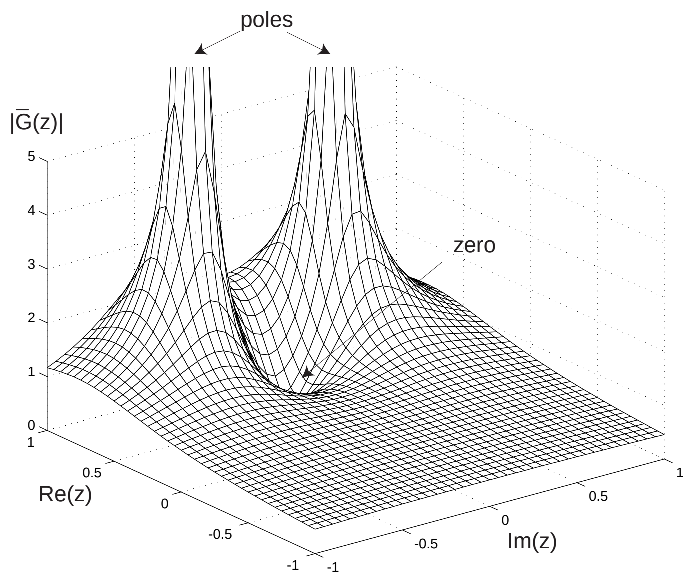



Figure 1.4: Illustration of poles and zeros when $A(\mathrm{q}^{-1}) = (1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1})$ and $B(\mathrm{q}^{-1}) = \mathrm{q}^{-1}(1 - z_1\mathrm{q}^{-1})$, with the poles $\lambda_1 = 0.7 + 0.4\mathrm{i}$, $\lambda_2 = 0.7 - 0.4\mathrm{i}$ and the zero $z_1 = 0.5$. These define $\bar{G}(z) = B(z)/A(z)$.

# Chapter 2

# Stability

Stability is the most important property and the least requirement on a control system. It means that if the system is left to itself, without input excitation, the output will settle to zero. This can be expressed as a condition that restricts the *poles* of the system to be inside the unit disk. There are also generalizations of the notion of stability which quantifies *how* the output settles to zero. The poles are then restricted further to a subset of the unit disk. It turns out that stability can easily be evaluated graphically by a frequency domain criterion. This is also used to derive the classical Nyquist criterion, which is a frequency domain criterion that *quantifies* stability margins, i.e. gives estimation of gain and phase variations of the plant with maintained stability of the closed loop. Frequency domain functions are also used for characterizing stationary responses for stable systems.

## 2.1 Stability definition and criteria

Consider the system with no input and arbitrary nonzero initial conditions

$$A(\mathrm{q}^{-1})y(k) = 0, \qquad k > 0 \tag{2.1}$$

**Definition 1 (Stability)** *The system (2.1) is stable if* $y(k) \to 0$*,* $k \to \infty$ *for all initial conditions* $y(0), \ldots, y(1 - \deg A)$.

To get criteria for stability it is instructive to first consider the simplest case, $A(\mathrm{q}^{-1}) = 1 - \lambda \mathrm{q}^{-1}$, for which the general solution to (2.1) is

$$y(k) = c\lambda^k$$

since $A(\mathrm{q}^{-1})y(k) = c\lambda^k - \lambda(c\lambda^{k-1}) = 0$ and $c = y(0)$ is an arbitrary initial condition. It is clear that the system is stable if and only if

$$|\lambda| < 1$$

Similarly, the general case with characteristic polynomial

$$A(\mathrm{q}^{-1}) = \prod_{i=1}^{m}(1 - \lambda_i \mathrm{q}^{-1})^{\nu_i}$$

can be derived (see Appendix)

$$A(\mathrm{q}^{-1})y(k) = 0, \quad \rightarrow y(k) = \sum_{i=1}^{m}[\sum_{j=0}^{\nu_i - 1} c_{ij}k^j]\lambda_i^k$$

where the coefficients $c_{ij}$ are complex. Collecting complex conjugate pairs, this can be rewritten with real parameters as

$$y(k) = \sum_{i=1}^{m_c}[\sum_{j=0}^{\eta_i - 1} k^j \rho_{ij} \sin(k \arg \lambda_i + \varphi_{ij})]|\lambda_i|^k$$

Just as in the simplest case above, the stability condition follows as

$$|\lambda_i| < 1, \forall i$$

**Example: 2.1**

Consider the difference equation

$$y(k) = 2y(k-1) - 0.99y(k-2), \qquad k > 0$$

Suppose first that the initial conditions are $y(-1) = 1$ and $y(0) = 0.9$. Then $y(k) \rightarrow 0$, $k \rightarrow \infty$ as illustrated in Fig. 2.1a. However, if one of the initial conditions are modified slightly to $y(-1) = 0.99$, the resulting solution grows to infinity, as shown in Fig. 2.1b. Clearly, the system is unstable since stability requires that $y(k) \rightarrow 0$, $k \rightarrow \infty$ **for all** initial conditions. Also, observe that that stability is an asymptotic property about what happens when time approaches infinity. Both solutions in Fig. 2.1a and b decay initially and it is therefore impossible to conclude that the system is stable or unstable by just looking at the first samples. The behavior can be understood by investigating the poles.

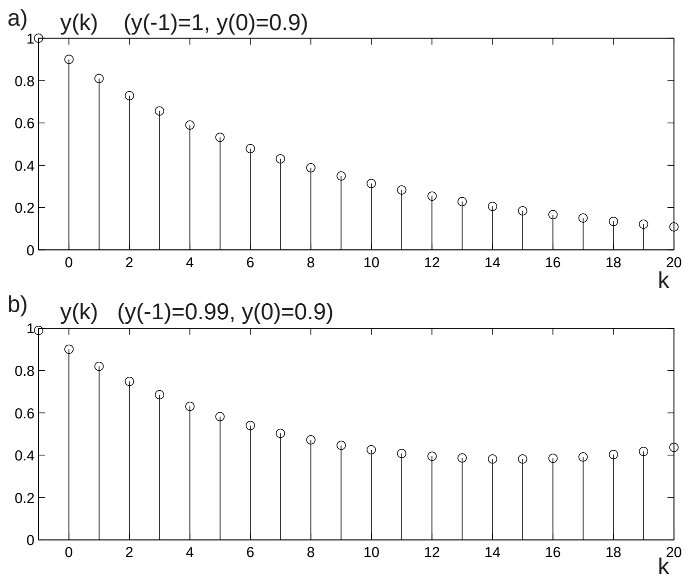



Figure 2.1: Example of an unstable system. **a)** Initial conditions chosen such that $y(k) \to 0$, $k \to \infty$. **b)** Slightly perturbation of the initial conditions in **a)**.

$$A(q^{-1}) = 1 - 2q^{-1} + 0.99q^{-2} = (1 - 0.9q^{-1})(1 - 1.1q^{-1})$$

The pole $\lambda_1 = 0.9$ is 'stable' while the pole $\lambda_2 = 1.1$ is 'unstable'. Since not all poles are inside the unit circle, the system is unstable. The solutions have the form

$$y(k) = c_1\lambda_1^k + c_2\lambda_2^k$$

By choosing $y(-1) = c_0$ and $y(0) = c_0\lambda_1$ (in the example above $c_0 = 1$) it follows that $c_1 = c_0\lambda_1$ and $c_2 = 0$. Since $c_2 = 0$ the unstable 'mode' is not seen in the solution in Fig. 2.1a. However, the slightly modified initial condition gives $c_2 \neq 0$ and the unstable part will eventually grow and dominate the solution as shown in Fig. 2.1b. □

## 2.2 Frequency domain criteria for stability

Stability can easily be displayed graphically by polar plots of frequency functions. The frequency function is constructed either from the characteristic polynomial or from the open-loop transfer function. The former results in the (modified) Mikhaylov criterion and the latter in the Nyquist criterion.

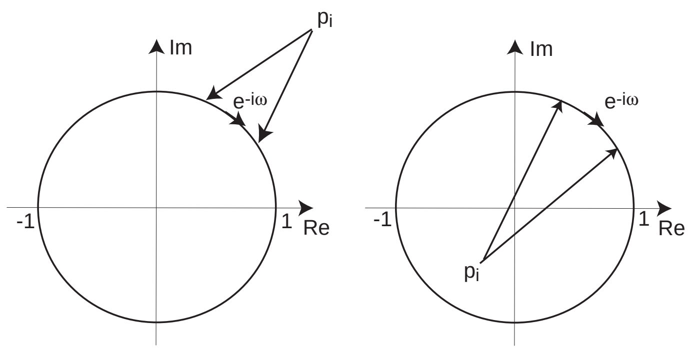



Figure 2.2: The change of argument of $\mathrm{e}^{-\mathrm{i}\omega} - p_i$ when $\omega$ goes from $0$ to $2\pi$. The total change is zero when $|p_i| > 1$ (left) and $-2\pi$ when $|p_i| < 1$ (right).

## The modified Mikhaylov criterion

Consider the polynomial

$$A(\mathrm{q}^{-1}) = a_0 + a_1\mathrm{q}^{-1} + \ldots + a_n q^{-n} = a_n \prod_{i=1}^{n} (\mathrm{q}^{-1} - p_i)$$

where $p_i$ are the roots, $A(p_i) = 0$, $i = 1, \ldots, n$. Evaluate the polynomial at the stability boundary, i.e. the unit circle $z = \mathrm{e}^{-\mathrm{i}\omega}$.

$$A(\mathrm{e}^{-\mathrm{i}\omega}) = a_n \prod_{i=1}^{n} (\mathrm{e}^{-\mathrm{i}\omega} - p_i)$$

Now, examine the total change of the argument of one factor $(\mathrm{e}^{-\mathrm{i}\omega} - p_i)$ when $\omega$ goes from zero to $2\pi$. Think of nailing one end of a rubber band to a stick that cannot turn and put the stick at $p_i$. Then stretch out the rubber band such that the other end is at $1$ and start to move this end clock-wise one revolution around the unit circle, see Fig. 2.2. The total change of argument is $-2\pi$ if $p_i$ is inside the unit circle, since the rubber band is then folded around the stick one revolution in the clock-wise sense. Conversely, if $p_i$ is outside the unit circle, the rubber band is not folded around the stick, but looks, after the movement around the unit circle, exactly as when we started. The change of argument is zero. Thus,

$$\Delta \arg_{0 \le \omega < 2\pi} (\mathrm{e}^{-\mathrm{i}\omega} - p_i) = \begin{cases} 0, & |p_i| > 1 \\ -2\pi, & |p_i| < 1 \end{cases}$$

and therefore

$$\Delta \arg_{0\leq\omega<2\pi} A(\mathrm{e}^{-\mathrm{i}\omega}) = \begin{cases} 0, & |p_i| > 1, \quad i = 1,\ldots,n \\ -2n\pi, & |p_i| < 1, \quad i = 1,\ldots,n \end{cases}$$

Hence, the necessary and sufficient condition for stability becomes

$$\Delta \arg_{0\leq\omega<2\pi} A(\mathrm{e}^{-\mathrm{i}\omega}) = 0$$

or differently stated:

**Theorem 1 (The modified Mikhaylov criterion)** *A system with the characteristic polynomial $A(\mathrm{q}^{-1})$ is stable if and only if $A(\mathrm{e}^{-\mathrm{i}\omega})$, $\omega = 0 \to 2\pi$ does not encircle the origin.*

**Example: 2.2**

Consider a stable characteristic polynomial

$$A(\mathrm{q}^{-1}) = 24 - 14\mathrm{q}^{-1} - \mathrm{q}^{-2} + \mathrm{q}^{-3}$$

The modified Mikhaylov locus $A(\mathrm{e}^{-\mathrm{i}\omega})$, $\omega = 0 \to 2\pi$, is shown in Fig. 2.3. From the locus in Fig. 2.3 it is directly seen that the system is stable since the origin is not enclosed.

□

## Necessary conditions for stability—easy to check first

Necessary conditions for stability, given a characteristic polynomial

$$A(\mathrm{q}^{-1}) = 1 + a_1\mathrm{q}^{-1} + \ldots + a_n\mathrm{q}^{-n}$$

are that

$$\begin{cases} \mathrm{sign}A(1) = \mathrm{sign}A(-1) \\ |a_n| < 1 \end{cases}$$

The first condition comes from the fact that, when the Mikhaylov locus $A(\mathrm{e}^{-\mathrm{i}\omega})$ is half way at $\omega = \pi$, the origin cannot be in between since then it will be enclosed. The second condition comes from the fact that $a_n = \lambda_1\lambda_2\ldots\lambda_n$ and $|\lambda_i| < 1, \forall i \Rightarrow |a_n| < 1$. These conditions are easily calculated by hand (head). If they are not satisfied, the system is unstable. Conversely, if they are satisfied,

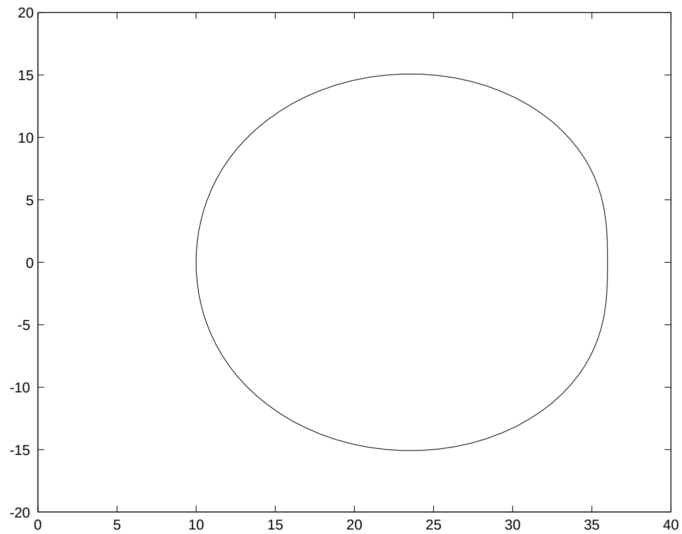



Figure 2.3: Modified Mikhaylov locus in Example 2.2.

more points $e^{-i\omega}$ (other than $\omega = 0$ and $\pi$) are needed to make a conclusion about stability. However, notice that for $n = 2$ the conditions

$$\begin{cases} \text{sign}A(1) = \text{sign}A(-1) > 0 \\ |a_2| < 1 \end{cases} \Leftrightarrow \begin{cases} 1 + a_1 + a_2 > 0 \\ 1 - a_1 + a_2 > 0 \\ a_2 < 1 \end{cases}$$

are also sufficient.

## The Nyquist criterion

Let $\frac{B}{A}$, $\frac{S}{R}$ denote the plant and the controller, respectively, and introduce the open-loop compensated system $G_{open} = \frac{BS}{AR}$. The following gives a relation between open-loop and closed-loop quantities ($A_c = A_{closed}$ for clarity):

$$1 + G_{open} = \frac{AR + BS}{AR} = \frac{A_{closed}}{A_{open}}$$

Now suppose the open-loop system $G_{open}$ has $N_u$ unstable poles. Then a direct application of the modified Mikhaylov criterion gives that the closed-loop system is stable conditioned that

$$
\begin{aligned}
0 &= \Delta \arg_{0\le\omega<2\pi} A_{closed}(e^{-i\omega}) \\
&= \Delta \arg_{0\le\omega<2\pi}[1+G_{open}(e^{-i\omega})] + \Delta \arg_{0\le\omega<2\pi} A_{open}(e^{-i\omega}) \\
&= \Delta \arg_{0\le\omega<2\pi}[1+G_{open}(e^{-i\omega})] - 2N_u\pi
\end{aligned}
$$

**Theorem 2 (The Nyquist criterion)** *An open-loop system $G_{open}(q^{-1})$ with $N_u$ unstable poles is stable in closed loop under -1 feedback if and only if $G_{open}(e^{-i\omega})$, $\omega = 0 \to 2\pi$ encircles $(-1, 0i)$ $N_u$ times in counter-clockwise direction.*

**Example: 2.3**

Consider the stable open-loop system

$$G_{open}(q^{-1}) = \frac{0.1q^{-2}}{(1-0.1q^{-1})(1-0.7q^{-1})(1-0.9q^{-1})}$$

The Nyquist locus is shown in Fig. 2.4. The closed-loop system is stable since there is no encirclement around -1. In fact, since the curve crosses the negative real axis at about $-1/3$, there is a *gain margin* of $3$; the possible increase of open-loop gain before the closed-loop becomes unstable. □

**Example: 2.4**

Consider the unstable open-loop system

$$G_{open}(q^{-1}) = \frac{0.1q^{-2}}{(1-0.1q^{-1})(1-0.7q^{-1})(1-1.1q^{-1})}$$

There is one unstable open-loop pole. The Nyquist locus in Fig. 2.5 shows that there is one encirclement of -1 in counter-clockwise sense. Therefore, the closed-loop system is stable. If the gain is increased about a factor of $2$ the curve will encircle -1 once in clockwise direction. Thus, the gain margin is about $2$. For larger gains there are two unstable closed-loop pole. □

## 2.3 Generalized stability and dominant poles

Stability of a system means that the output settles to zero when the system is left alone without excitation. But the stability concept gives no quantification of how

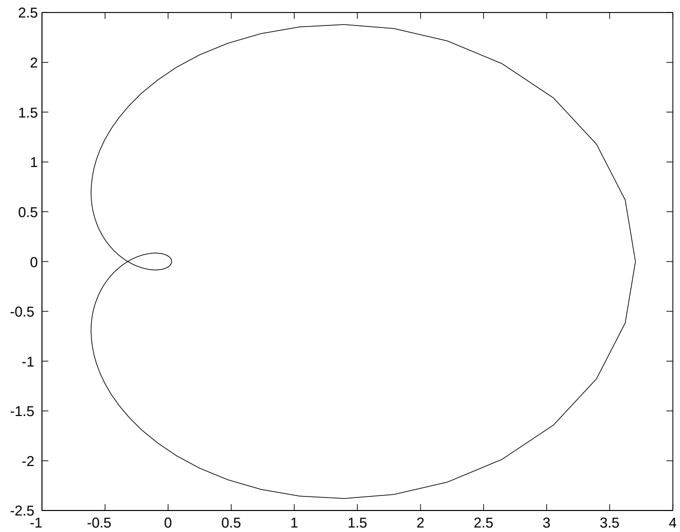



Figure 2.4: The Nyquist locus for the stable system in Example 2.3. The closed-loop system is stable since there is no encirclement of $(-1, 0i)$, and the gain margin is about $3$.

it settles to zero. From a given initial condition the response can be oscillatory, monotonous, fast or slow while approaching zero. In all these cases the systems are stable but from a practical point of view very different. Since stability restricts the pole location in the complex plane a natural extension to quantify the output settlement would be to further restrict the pole location. This is, however, not as obvious as it might seem since the transient behavior we like to quantify also depends on the number of poles and not only on their locations. Also, of course, the transient depends on the initial conditions and zeros. Let us anyway consider a system with one complex conjugate pole-pair.

$$(1-\lambda \mathrm{q}^{-1})(1-\bar{\lambda}\mathrm{q}^{-1})y(t) = b\mathrm{q}^{-1}u(t)$$

with $b = (1-\lambda)(1-\bar{\lambda})$ such that the steady-state gain is one. Since there is no zero that can influence the transient we can equally well consider the step response. Thus, let the initial conditions be $y(0) = y(-1) = 0$ and $u(t) = 1$, $t \geq 0$. The restriction of the poles inside the unit circle that corresponds to an overshoot of $50\%$ is shown in Fig. 2.6. For pole locations inside the $50\%$-overshoot level curve the overshoot is smaller. Another level curve in Fig. 2.6 shows the pole locations that give the same speed ($y(1)$ are all the same). For pole locations to

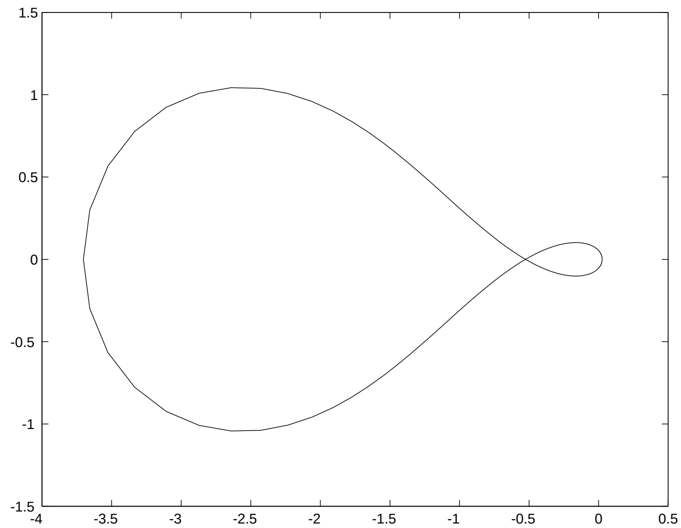



Figure 2.5: The Nyquist locus for the unstable system in Example 2.4. The closed-loop system is stable since there is one counter-clockwise encirclement of $(-1, 0i)$.

the left of that curve the responses are faster. This exercise indicates that a reasonable pole placement should be such that $\lambda_i$, $\forall i$ are inside the right half of the unit circle; closer to zero for faster and closer to 1 for slower response. Denote $D$ the restricted domain inside the unit circle where the $\lambda_i$ should be located. Generalized stability then requires all $\lambda_i$ to be inside $D$. Parameterize the boundary of $D$ as $r(\omega)\mathrm{e}^{\mathrm{i}\omega}$ where $r(\omega)$ shrinks the unit circle appropriately, e.g. as in Fig. 2.6. It is then straightforward to evaluate generalized stability using the modified Mikhaylov criterion after replacing $\mathrm{q}$ by $r(\omega)\mathrm{e}^{\mathrm{i}\omega}$ instead of $\mathrm{e}^{\mathrm{i}\omega}$.

**Theorem 3 (The modified Mikhaylov criterion: generalized stability)** *A system with the characteristic polynomial $A(\mathrm{q}^{-1})$ is $D$-stable if and only if $A(r^{-1}(\omega)\mathrm{e}^{-\mathrm{i}\omega})$, $\omega = 0 \rightarrow 2\pi$ does not encircle the origin.*

The $D$-region can be defined as

$$r(\omega) = \begin{cases} \mathrm{e}^{-\sqrt{\omega_{\min}^2 - \omega^2}} & 0 \leq \omega \leq \omega_{\min}\sqrt{1-\zeta_{\min}^2} \\ \mathrm{e}^{-\frac{\zeta_{\min}}{\sqrt{1-\zeta_{\min}^2}}\omega} & \omega_{\min}\sqrt{1-\zeta_{\min}^2} < \omega \leq \pi \end{cases}$$

which is derived from sampling of a continuous-time system with the poles at the smallest radius $\omega_{\min}$ from origin and with angles to the negative real axis

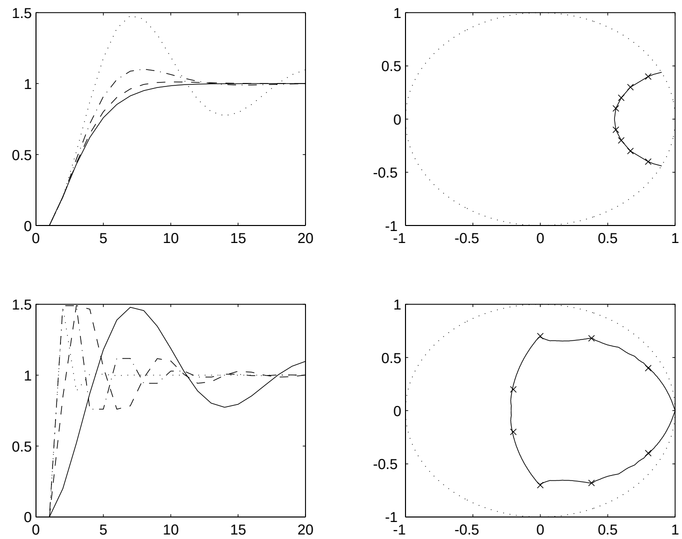



Figure 2.6: Right: pole locations ('x'); left: corresponding step responses.

inferior to $\arccos \zeta_{\min}$. The former bound certifies a minimal speed and the latter a maximal overshoot. With $\omega_{\min} = 0.5$ and $\zeta_{\min} = 0.25$, this $D$-region is close to the combination of the $D$-regions shown in Fig. 2.6, but only for small $\omega$. The differences are due to a zero appearing after sampling whose effect mainly influences the transients at high frequencies.

The introduction of $D$-stability based on a system having only one complex pole-pair might seem a bit naive. When there are more poles the transients are different anyway and the $D$-region becomes meaningless. Conversely, the study of $D$-stability has taught us reasonable pole placement for the *dominant poles*. The dominant poles are the poles that determine the slow response of the system. This is illustrated in the following example.

**Example: 2.5**

Consider the two systems

$$G_1 = \frac{b_1}{(1-\lambda_1 q^{-1})(1-\bar{\lambda}_1 q^{-1})}$$
$$G_2 = G_1 \frac{b_2}{(1-\lambda_2 q^{-1})(1-\bar{\lambda}_2 q^{-1})(1-\lambda_3 q^{-1})(1-\bar{\lambda}_3 q^{-1})}$$

where $\lambda_1 = 0.9 + 0.2\mathrm{i}$, $\lambda_2 = -0.3 + 0.7\mathrm{i}$, $\lambda_3 = -0.7 + 0.5\mathrm{i}$ and $b_1$, $b_2$ chosen to make both system have steady-state gain $1$. The pole $\lambda_1$ is responsible for the slow response and dominates the behavior of the step response (Fig. 2.7). □

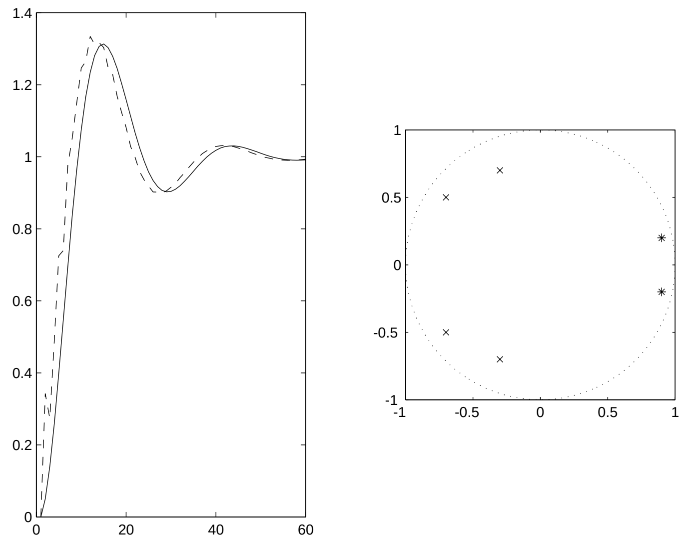



Figure 2.7: Example 2.5. Left: step responses of the systems $G_1$ (solid) and $G_2$ (dashed). Right: the poles $\lambda_i$ of $G_1$ ('+') and $G_2$ ('x').

## 2.4 Stationary responses

It is often of interest to calculate the *stationary* or *forced* response. This is the remaining response after a transient. Conditioned, of course, that the transient caused by initial conditions fades out, i.e. that the system is stable.

### Steady-state gain

Consider the system $y(k) = G(\mathrm{q}^{-1})u(k)$ where $G(\mathrm{q}^{-1})$ is a stable transfer operator and the input is a unit step

$$\begin{cases} u(k) = 1, & k \geq 0 \\ u(k) = \frac{1}{1-\mathrm{q}^{-1}}\varepsilon(k) & \end{cases} \qquad \text{where } \varepsilon(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

where $\varepsilon(k)$ is a unit pulse. The two expression above are equivalent but the second one is in this context more convenient to use. After rewriting it as a difference equation, it is easy to check that this is, indeed, a unit step.

$$\begin{aligned}
&u(k) = u(k-1) + \varepsilon(k) \\
&u(0) = u(-1) + \varepsilon(0) = 0 + 1 = 1 \\
&u(1) = u(0) + \varepsilon(1) = 1 + 0 = 1 \\
&u(2) = u(1) + \varepsilon(2) = 1 + 0 = 1 \\
&\vdots
\end{aligned}$$

The response can now be written as

$$y(k) = G(\mathrm{q}^{-1})\frac{1}{1-\mathrm{q}^{-1}}\varepsilon(k) = G(1)\frac{1}{1-\mathrm{q}^{-1}}\varepsilon(k) + \underbrace{[G(\mathrm{q}^{-1}) - G(1)]\frac{1}{1-\mathrm{q}^{-1}}\varepsilon(k)}_{\to 0}$$

The second term on the right hand side is stable, since $G$ is stable and since the unstable factor $1-\mathrm{q}^{-1}$ is canceled. Thus, this is the transient response that fades out. The remaining stationary response is

$$y(k) \to G(1)$$

for large $k$. This is called the *steady-state gain*.

## Frequency response

The stationary frequency response to the input $u(k) = \cos(\omega k + \theta)$ can be derived in a similar fashion by introducing the *complex* signal

$$\begin{cases} u^*(k) = \mathrm{e}^{\mathrm{i}(\omega k+\theta)}, & k \geq 0 \\ u^*(k) = \frac{e^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k) \end{cases}$$

where the real signal is $u(k) = \mathrm{Re}[u^*(k)] = \cos(\omega k + \theta)$. Notice that the unit step described above corresponds to $\omega = 0$ (and $\theta = 0$). As before we can write the (now complex) response $y^*$ in two terms, one giving the stationary, remaining response and a second giving a transient (for a stable system)

$$\begin{aligned}
y^*(k) = G(\mathrm{q}^{-1})\tfrac{e^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k) = \quad & G(\mathrm{e}^{-\mathrm{i}\omega})\tfrac{e^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k) \\
& + \underbrace{[G(\mathrm{q}^{-1}) - G(\mathrm{e}^{-\mathrm{i}\omega})]\frac{e^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k)}_{\to 0}
\end{aligned}$$

Taking the real part, the remaining stationary response is

$$\begin{aligned}
y(k) \to \quad & \mathrm{Re}\, G(\mathrm{e}^{-\mathrm{i}\omega})\tfrac{e^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k) = |G(\mathrm{e}^{-\mathrm{i}\omega})|\mathrm{Re}\tfrac{\mathrm{e}^{\mathrm{i}(\theta+\arg G(\mathrm{e}^{-\mathrm{i}\omega}))}}{1-\mathrm{e}^{\mathrm{i}\omega}\mathrm{q}^{-1}}\varepsilon(k) = \\
& |G(\mathrm{e}^{-\mathrm{i}\omega})|\cos(\omega k + \theta + \arg G(\mathrm{e}^{-\mathrm{i}\omega}))
\end{aligned}$$

# Appendix

The general solution to the homogenous difference equation $A(\mathrm{q}^{-1})y(k) = 0$ is successively derived below.

**Distinct $n$ real poles:** Let $A = \prod_{i=1}^{n} A_i(\mathrm{q}^{-1}) = \prod_{i=1}^{n}(1 - \lambda_i \mathrm{q}^{-1})$ with $\lambda_i \neq \lambda_j$, for $i \neq j$. As above, it holds that $A_i(\mathrm{q}^{-1})\lambda_i^k = 0$. Consequently, it also holds that $A(\mathrm{q}^{-1})y(k) = 0$ with

$$y(k) = \sum_{i=1}^{n} c_i \lambda_i^k \tag{2.2}$$

It remains to check that the (real) coefficients $c_i$, $i = 1, \ldots, n$ can be solved uniquely from the initial conditions. Applying (2.2) successively it follows that

$$\begin{pmatrix} y(0) \\ y(-1) \\ \vdots \\ y(1-n) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \ldots & \lambda_n^{-1} \\ \vdots & & & \vdots \\ \lambda_1^{1-n} & & & \lambda_n^{1-n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The matrix above has full rank since the roots are all different. The coefficients can therefore be calculated uniquely which proofs that the general solution to (2.1) is given by (2.2). Thus, it follows that a system with distinct real poles is stable if and only if

$$|\lambda_i| < 1, \forall i$$

**Multiple $\nu$ real poles:** As above let $A_1(\mathrm{q}^{-1}) = 1 - \lambda_1 \mathrm{q}^{-1}$ and let $A = A_1^{\nu}$. Then notice that

$$A_1 k^m \lambda_1^k = \lambda_1^k (m k^{m-1} + \ldots)$$

where the expression in the parenthesis is a polynomial in $k$ of order $m - 1$. Thus, the solution (2.1) is

$$y(k) = (\sum_{j=0}^{\nu-1} c_j k^j)\lambda_1^k \tag{2.3}$$

since $A(\mathrm{q}^{-1})y(k) = A_1^{\nu} y(k) = 0$. It is straightforward to check that the coefficients $c_j$ can be uniquely calculated as well. The solution (2.3) consists of terms of the type $k^m \lambda^k$ which converge to zero if $|\lambda| < 1$ irrespective of the diverging polynomial $k^m$. This can intuitively be realized by regarding the multi-pole system as several one-pole systems in series, i.e. $u_{i+1} = y_i$, $A_1 y_i = u_i$, $i = 1, \ldots, n$,

$u = u_1 = 0$, $y = y_n$. Since all subsystems are stable, $y_i \to 0$, $k \to \infty$, $\forall i$. More formally, express $k^m$ in the basis of $\lambda$ as $k^m = \lambda^{m\ {}^{\lambda}\!\log k}$. Then $k^m\lambda^k = \lambda^{f(k)}$ with $f(k) = m\ {}^{\lambda}\!\log k + k$. If $\lambda < 1$ then there exists an $\alpha > 0$ such that $e = \lambda^{-\alpha}$ which gives ${}^{\lambda}\!\log e = -\alpha$. Since $k = e^{\ln k}$ it follows that ${}^{\lambda}\!\log k = \ln k\ {}^{\lambda}\!\log e = -\alpha \ln k$ which gives $f(k) = k - \alpha m \ln k \to \infty$, $k \to \infty$. Hence, $k^m\lambda^k = \lambda^{f(k)} \to 0$, $k \to \infty$, if $|\lambda| < 1$.

The general solution to (2.1) when

$$A(\mathrm{q}^{-1}) = \prod_{i=1}^{m}(1 - \lambda_i \mathrm{q}^{-1})^{\nu_i}$$

where $\deg A = \sum_{i=1}^{m} \nu_i$ is

$$y(k) = \sum_{i=1}^{m}[\sum_{j=0}^{\nu_i-1} c_{ij}k^j]\lambda_i^k \tag{2.4}$$

and $y(k) \to 0$, $k \to \infty$ if and only if $|\lambda_i| < 1$, $\forall i$.

**Complex-conjugate poles:** When the poles are complex, the above solutions still hold. But the formulas are somewhat inconvenient since the coefficients $c_i$ now become complex. To avoid complex coefficients, consider two complex conjugated pairs at a time, $c_i\lambda_i^k + \bar{c}_i\bar{\lambda}_i^k$. Introduce $c_i = \rho_i e^{j\varphi_i}/(2j)$, with $\rho_i$ real, to get $c_i\lambda_i^k + \bar{c}_i\bar{\lambda}_i^k = |\lambda_i|^k \rho_i \sin(\arg\lambda_i k + \varphi_i)$.

Let $A$ have $m_r$ real roots and $m_c$ complex conjugate pair of roots with multiplicity $\nu_i$ and $\eta_i$, respectively. Then (2.4) can be rewritten in real coefficients as

$$y(k) = \sum_{i=1}^{m_r}[\sum_{j=0}^{\nu_i-1} k^j c_{ij}]\lambda_i^k = \sum_{i=1}^{m_c}[\sum_{j=0}^{\eta_i-1} k^j \rho_{ij}\sin(k\arg\lambda_i + \varphi_{ij})]|\lambda_i|^k$$

# Chapter 3

# Structures of control systems

The control structure in Fig. 1.3 describes a two-degrees-of-freedom controller. Two transfer functions, $G_r(\mathrm{q}^{-1})$ and $G_y(\mathrm{q}^{-1})$, are chosen. This is the general structure and all other control structures of linear time-invariant controllers can be represented in this form. With *structure*, we simply mean the parameterization giving the block-diagram configuration. Different structures have been proposed because they make the design of certain objectives more straightforward. It should be noted, though, that one structure does not allow better control than others, since they all can be represented as in Fig. 1.3.

## 3.1 A simple structure: one-degree-of-freedom

If $G_r(\mathrm{q}^{-1}) = G_y(\mathrm{q}^{-1}) = G_e(\mathrm{q}^{-1})$ the structure becomes simplified, see Fig. 3.1. In this case only $G_e(\mathrm{q}^{-1})$ is chosen. This clearly restricts the possible designs of

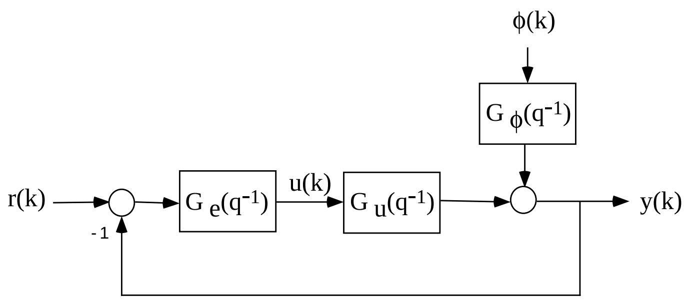



Figure 3.1: A one-degree-of-freedom structure.

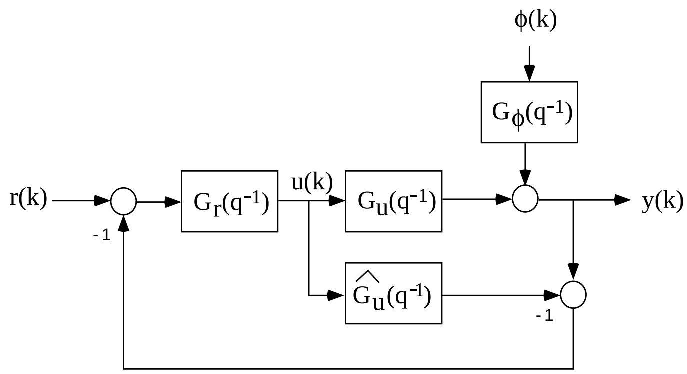



Figure 3.2: Internal model control structure.

$H_r(\mathrm{q}^{-1})$. But it does not restrict the design of $H_\varphi(\mathrm{q}^{-1})$ and is therefore enough for the regulator problem, i.e. when only the disturbance rejection is considered.

## 3.2 Internal model control

In a model-based design, the controller is a function of the plant model. Sometimes, the controller parameterization is such that the full plant model is in one block in the structure. This is called *internal model control*. The internal model control design is popular basically because of its simple and intuitive design philosophy. The structure is shown in Fig. 3.2. The controlled output is given by

$$y(k) = \frac{G_u G_r}{1 + G_r[G_u - \hat{G}_u]} r(k) + \frac{[1 - G_r \hat{G}_u] G_\varphi}{1 + G_r[G_u - \hat{G}_u]} \varphi(k)$$

Notice, that one block contains a model, $\hat{G}_u(\mathrm{q}^{-1})$, of the plant, $G_u(\mathrm{q}^{-1})$. If the model is perfect, i.e. $\hat{G}_u(\mathrm{q}^{-1}) = G_u(\mathrm{q}^{-1})$, the feedback becomes independent of the process output and contains only the disturbance. Thus, the system appears to be in open loop with another filtering of the disturbance:

$$y(k) = G_u G_r r(k) + [1 - G_u G_r] G_\varphi \varphi(k) \tag{3.1}$$

The design philosophy using the internal model control structure is evident from (3.1): choose $G_r$ such that $\hat{G}_u G_r \approx 1$ with as good approximation as possible in a desired frequency range up to a chosen bandwidth. Notice, that the choice $G_r = 1/\hat{G}_u$ is impossible since the plant has always at least one time delay and the

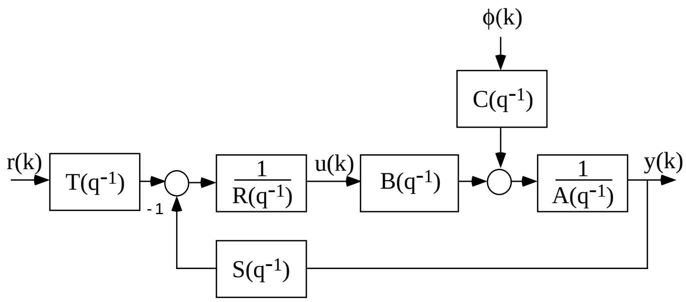



Figure 3.3: A general canonical structure.

controller must be causal. The corresponding one-degree-of-freedom controller is

$$G_e = \frac{G_r}{1 - G_r \hat{G}_u}$$

Clearly, the poles of $\hat{G}_u(\mathrm{q}^{-1})$ are zeros of $G_e(\mathrm{q}^{-1})$. Thus, the controller cancels all process poles. This controller structure can therefore only be used for stable plants. To realize this, let the disturbance enter at the process input, i.e. $G_\varphi(\mathrm{q}^{-1}) = G_u(\mathrm{q}^{-1})$. Then it is clear from (3.1) that if the system is open-loop unstable it will be closed-loop unstable, irrespective of the choice of $G_r(\mathrm{q}^{-1})$. Internal model control is also inappropriate for stable systems having oscillatory poles since these maintain after feedback. This is a consequent of the cancellation of all process poles by the controller.

## 3.3 A general canonical structure

It is important to define a *canonical structure* for our control scheme. That means that each element in the block diagram contains a polynomial. In addition, each polynomial is unique and does not appear in more than one block. The structure is shown in Fig. 3.3. This is a two-degree-of-freedom control structure equivalent to the one in Fig. 1.3, but where the common denominator $R(\mathrm{q}^{-1})$ in $G_r(\mathrm{q}^{-1})$ and $G_y(\mathrm{q}^{-1})$ is placed in a separate block. Notice, that since the backward-shift operator $\mathrm{q}^{-1}$ is used, it is possible to move around numerators and denominators like this and still each block is realizable, i.e. all signals between the blocks in Fig. 3.3 can be generated. It would not have been the case if we had chosen polynomials in the forward-shift operator (or differential operator for continuous systems). There are clearly advantages with using the backward-shift operator

$\mathrm{q}^{-1}$. All signals in the block diagram can be generated and no special care needs to be taken about the relative degree between polynomials for ensuring causality.

## 3.4 Pole-zero cancellations

Even though the output $y(k)$ usually is of the primary interest it is sometimes important to also consider the behavior of the system input $u(k)$. Regarding the closed-loop system in Fig. 3.3 as a system with two external inputs $r(k)$, $\varphi(k)$ and with the *outputs* $y(k)$, $u(k)$, it follows

$$\begin{aligned} y &= \frac{BT}{A_c} r + \frac{RC}{A_c} \varphi \\ u &= \frac{AT}{A_c} r - \frac{SC}{A_c} \varphi \end{aligned}$$

where $A_c = AR + BS$. The closed-loop system is stable if and only if $A_c$ is a 'stable' polynomial, i.e. all poles satisfy $|\lambda_i| < 1$ where $A_c(\lambda_i^{-1}) = 0$. This is important to keep in mind, especially when only one of the four transfer operators is considered in the design. When only the response from $r$ to $y$ is considered it is tempting to choose the controller such that a pole/zero cancellation occurs between $A_c$ and $B$. This is disastrous if the poles and zeros that cancel are unstable since the same cancellation of the unstable poles in $A_c$ is not taken place in the other three transfer operators. The system is in such a case unstable despite that the response between $r$ and $y$ is stable. Not even in the noise-free case one can allow unstable pole/zero cancellations since the response from $r$ to $u$ becomes unstable. This is due to the fact that $A$ and $B$ are coprime and consequently cannot both cancel the same poles in $A_c$. Always when there is a pole-zero cancellation in a feedback loop the characteristic polynomial $A_c$ becomes invariant with respect to the canceled poles. This is important to be aware of, since strategies that encourage cancellations, like internal model control, are disastrous when the canceled poles are unstable.

Although unstable pole-zero cancellation within the loop should be avoided there are other situation when it is useful. Consider, for example, the step disturbance

$$\varphi(k) = \frac{1}{D(\mathrm{q}^{-1})} \varepsilon(k)$$

where $\varepsilon(k)$ is a unit pulse and $D(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-1}$. With $r = 0$, the system output and input are governed by the dynamics

$$\begin{aligned} y &= \frac{RC}{A_c D} \varepsilon \\ u &= -\frac{SC}{A_c D} \varepsilon \end{aligned}$$

Even though $A_c$ is stable, both responses, $y$ and $u$ are unstable unless there is an unstable pole-zero cancellation that removes the instability caused by $D(\mathrm{q}^{-1})$. Since the major concern is the output, we choose to cancel $D$ by including it in $R$, i.e. $R = R_1 D$. This unstable pole-zero cancellation eliminates the disturbance from the output $y$ asymptotically. The resulting controller is, loosely speaking, said to have integral action since the factor $1/(1-\mathrm{q}^{-1})$ can be viewed as a discrete-time integrator (more precisely, summator). This is a typical case and many controllers have integral action.

We have seen that unstable pole-zero cancellation is important and can be both disastrous, when it occurs within the loop, or advantageous, when it occurs between the closed-loop output-disturbance response model and the model of the disturbance itself. The structure of a control system should be helpful in the controller design and rather than suggesting disastrous cancellations support the cancellations that are indispensable.

## 3.5 PID controller discretization

A PID controller is often described in textbooks as

$$u(t) = P(t) + I(t) + D(t) = [K(e(t)] + [\frac{K}{T_i}\int_0^t e(s)ds] + [KT_d\frac{d}{dt}e(t)]$$

where $e(t) = r(t) - y(t)$. However, the derivative is not implemented like this. Since $r$ can be discontinuous, derivation of $r$ should be avoided. Also, derivations increase noise sensitivity. The derivative is therefore implemented as a high pass filter according to

$$\frac{T_d}{N}\frac{dD}{dt} + D = -KT_d\frac{dy}{dt}$$

To implement a continuous-time control law such as a PID controller on a digital computer, it is necessary to approximate the derivatives (the integral part can be written $\frac{dI}{dt} = \frac{K}{T_i}e$). This can be done in many ways. One possibility is to use *backward difference*:

$$\frac{d}{dt} \rightarrow \frac{1-\mathrm{q}^{-1}}{h}$$

where $h$ is the sampling period. By discretizing the modified PID controller above using backward difference approximation, the controller can be written

$$R(\mathrm{q}^{-1})u(k) = -S(\mathrm{q}^{-1})y(k) + T(\mathrm{q}^{-1})r(k)$$

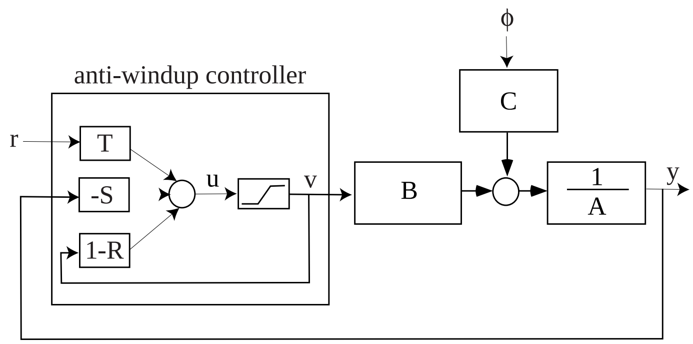



Figure 3.4: An anti-windup implementation.

with

$$R(q^{-1}) = (1 - q^{-1})(1 - aq^{-1})$$
$$S(q^{-1}) = s_0 + s_1 q^{-1} + s_2 q^{-2}$$
$$T(q^{-1}) = t_0 + t_1 q^{-1} + t_2 q^{-2}$$

It is straightforward to show that $a = T_d/(T_d + Nh)$ and

$$\begin{cases} s_0 = K(1 + h/T_i + Na) \\ s_1 = -K(1 + a[1 + h/T_i + 2N]) \\ s_2 = Ka(1 + N) \end{cases} \qquad \begin{cases} t_0 = K(1 + h/T_i) \\ t_1 = -K(1 + a[1 + h/T_i]) \\ t_2 = Ka \end{cases}$$

## 3.6 Anti-windup

In most practical controller implementations, the control signal is limited by hardware. Also, many controllers have integral action, i.e. they are unstable in open loop. This means that once the actual control signal saturates, the loop is open and the calculated control signal keeps increasing. This is called *integrator windup*. Then whenever the control error changes sign, it will take a long time to wind back the integrator from the enormous value it had grown to. This causes nasty oscillations. One way to get around this problem is to change the controller according to

$$u(k) = -Sy(k) + Tr(k) + (1 - R)v(k)$$
$$v(k) = \text{sat}[u(k)]$$

If the plant is stable $y(k)$ is bounded, $r(k)$ is assumed to be a bounded reference and $v(k)$ is bounded by the saturation. Therefore, also $u(k)$ must be bounded and windup is avoided. The construction is shown in Fig. 5.5.

# Chapter 4

# Pole-placement Design

Feedback is the most important aspect of control. By feedback, unstable systems can be stabilized or damping of oscillatory systems can be improved. Unmeasurable disturbances can be rejected and the closed-loop system can be made insensitive to changes of plant dynamics. In this chapter it is described how to calculate a solution to the feedback problem. Algorithmic aspects and tricks for hand calculations are given. Also, it is shown how to select particular solutions for disturbance rejection and noise sensitivity reduction.

## 4.1 The polynomial equation

The feedback polynomials $R$ and $S$ should be solved from the polynomial equation

$$AR + BS = A_c \tag{4.1}$$

This equation has many names: it is sometimes called the Diophantine equation, which refers to the Greek mathematician Diophantus (who by the way never studied this equation); sometimes it is referred to as the Bezout identity; according to Vidyasagar it should refer to a Hindu with the name Ariabhatta (born 476) who was the first mathematician of antiquity to formulate and find *all* solutions to this equation. Without any preferences one could adopt the abbreviation DAB, standing for Diophantus-Ariabhatta-Bezout (and not for the german beer). Conversely, the ancient greek and hindu did not consider an equation of polynomials, but of integers, which can be treated in a similar way from an algebraic point of view. Therefore, and in order to avoid the debate, we simply choose the notation:

*polynomial equation* in $R$ and $S$, i.e $A$, $B$ and $A_c$ are given and $R$, $S$ are to be calculated.

## How to calculate a solution

### Cancel common factors

In a minimal plant model description $A$ and $B$ are coprime, i.e. they have no common factors. Generally, $A_c$ might include factors of $A$ and $B$. Suppose for example that a factor $A_1$ of $A = A_1A_2$ is included as $A_c = A_{c1}A_1$. Then it follows that the same factor $A_1$ necessarily must be in $S = S_1A_1$, since $BS = A_c - AR = A_1(A_{c1} - A_2R)$ and $B$ does not include $A_1$ by the coprimeness assumption. But then the factor $A_1$ is included in all three terms and it can be canceled from the equation, resulting in another polynomial equation $A_2R + BS_1 = A_{c1}$. Now, in this equation there are no common factors between $A_2$, $B$ and $A_{c1}$,. If $A_c$ includes factors of $B$, these can be canceled similarly. Once the solution has been found from the modified polynomial equation, the canceled factors can be multiplied afterwards. It is therefore no restriction to assume that $A$, $B$ and $A_c$ all are coprime when deriving the general solution to (4.1).

### Write an equation system

The polynomial equation (4.1) can also be written as an equation system by identifying the coefficients of equal powers of $\mathrm{q}^{-1}$:

$$\sum_{i+j=k} a_i r_j + \sum_{i+j=k} b_i s_j = a_{c_k}, \qquad k = 0, \ldots, n$$

where $n = \max\deg[AR, BS]$, or in matrix form

$$n+1 \left\{ \vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}} \right. \underbrace{\begin{pmatrix} a_0 & \ldots & 0 \\ a_1 & \ddots & \vdots \\ a_2 & \ddots & a_0 \\ \vdots & \ddots & \vdots \end{pmatrix.}_{\deg R+1} \!\!\!\! \underbrace{\begin{matrix} b_0 & \ldots & 0 \\ b_1 & \ddots & \vdots \\ b_2 & \ddots & b_0 \\ \vdots & \ddots & \vdots \end{matrix}\Bigg)}_{\deg S+1} \begin{pmatrix} r_0 \\ \vdots \\ s_0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{c_0} \\ a_{c_1} \\ \vdots \\ a_{c_n} \end{pmatrix}$$

The matrix is invertible if there are as many independent equations as unknown, i.e. if

$$n + 1 = \deg R + \deg S + 2 \tag{4.2}$$

Since $n = \max \deg[AR, BS]$ this gives two possibilities. Suppose $\max \deg[AR, BS] = \deg AR$. Then it follows from (4.2) that $\deg S = \deg A - 1$ and consequently $\deg AR = \deg(A_c - BS) = \max \deg[A_c, BS]$, which after subtraction of $\deg A$ results in $\deg R = \max[\deg B - 1, \deg A_c - \deg A]$. Thus, one solution exists with

$$\begin{cases} \deg S = \deg A - 1 \\ \deg R = \max[\deg B - 1, \deg A_c - \deg A] \end{cases}$$

The symmetry gives the other solution after substituting $S$, $A$ and $B$ for $R$, $B$ and $A$, respectively.

$$\begin{cases} \deg R = \deg B - 1 \\ \deg S = \max[\deg A - 1, \deg A_c - \deg B] \end{cases}$$

The two solutions coincide if $\deg A_c \leq \deg A + \deg B - 1$. Since $A_c$ is our design choice we can always make this upper degree bound satisfied. Thus, a standard choice is

$$\begin{cases} \deg A_c \leq \deg A + \deg B - 1 \\ \deg R = \deg B - 1 \\ \deg S = \deg A - 1 \end{cases} \tag{4.3}$$

**Example: 4.1**

Let the plant model be $A = 1 + a_1 \mathrm{q}^{-1} + a_2 \mathrm{q}^{-2}$, $B = b_1 \mathrm{q}^{-1} + b_2 \mathrm{q}^{-2}$ and the desired characteristic polynomial $A_c = 1 + a_{c1} \mathrm{q}^{-1} + a_{c2} \mathrm{q}^{-2} + a_{c3} \mathrm{q}^{-3}$. From (4.3), $R = 1 + r_1 \mathrm{q}^{-1}$ and $S = s_0 + s_1 \mathrm{q}^{-1}$. Identifying coefficients of equal power of $\mathrm{q}^{-1}$ in (4.1) gives the equation system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_2 & a_1 & b_2 & b_1 \\ 0 & a_2 & 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 \\ r_1 \\ s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{c1} \\ a_{c1} \\ a_{c_3} \end{pmatrix}$$

If a computer program like Matlab is available the solution is easily found. However, for hand calculation the setup above is inconvenient. One obvious simplification is to eliminate the first raw. This gives

$$\begin{pmatrix} 1 & b_1 & 0 \\ a_1 & b_2 & b_1 \\ a_2 & 0 & b_2 \end{pmatrix} \begin{pmatrix} r_1 \\ s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} a_{c1} - a_1 \\ a_{c1} - a_2 \\ a_{c_3} \end{pmatrix}$$

However, for hand calculations another selection of independent equations is sometimes more convenient. Find $p_1$, $p_2$, and $z_1$ satisfying $A(p_1) = A(p_2) = B(z_1) = 0$. Evaluating (4.1) at $p_1$, $p_2$ and $z_1$ gives three independent equations (if $p_1 \neq p_2$). This results in

$$\left.\begin{array}{r} B(p_1)S(p_1) = A_c(p_1) \\ B(p_2)S(p_2) = A_c(p_2) \end{array}\right\} \rightarrow s_0, s_1$$
$$A(z_1)R(z_1) = A_c(z_1) \qquad \rightarrow r_1$$

□

## Pre-chosen fixed factors in $R$ and $S$

Not only $A_c$ is of interest but also the polynomials $R$ and $S$ themselves. It is often not a minimal degree solution that is sought but a solution with more parameters allowing also other constraints to be taken into account. Reconsider both the system output and input:

$$y = \frac{BT}{A_c}r + \frac{RC}{A_c}\varphi$$
$$u = \frac{AT}{A_c}r - \frac{SC}{A_c}\varphi$$

From the reference point of view, the particular solution $R$ and $S$ for a given $A_c$ is irrelevant. From the disturbance point of view, however, $R$ influences the response from $\varphi$ to $y$ and $S$ influences the response from $\varphi$ to $u$. It is therefore important to be able to solve the polynomial equation in such a way that the solutions give fixed pre-designed factors in $R$ and $S$. For example, suppose $\varphi$ is a constant disturbance $\varphi(k) = \varphi(k-1)$. Then, the factor $R_f(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-1}$ makes $R_f(\mathrm{q}^{-1})\varphi(k) = 0$. Another example, suppose $\varphi$ represents measurement noise, i.e. $y$ is the system output plus measurement noise. Then, it is the system output rather than $y$ we like to control. In such a situation, it is desirable not to eliminate the noise in $y$ but instead eliminate it from $u$ which excite the system via $\varphi$. With e.g. an oscillating noise as $\varphi(k) = -\varphi(k-1)$ we would choose $S_f(\mathrm{q}^{-1}) = 1 + \mathrm{q}^{-1}$ such that $S_f(\mathrm{q}^{-1})\varphi(k) = 0$.

Pre-chosen factors, $R_f$ and $S_f$, can easily be included in the pole-placement problem simply by first associating them to $A$ and $B$ when solving the polynomial equation. Thus, first let $A' = AR_f$, $B' = BS_f$ and choose $\deg A_c \leq \deg A' + \deg B' - 1$. Then solve the polynomial equation

$$A'R_1 + B'S_1 = A_c$$

with respect to $R_1$ and $S_1$, where

$$\begin{cases} \deg R_1 = \deg B' - 1 \\ \deg S_1 = \deg A' - 1 \end{cases}$$

Afterwards, calculate the controller polynomials

$$R = R_f R_1$$
$$S = S_f S_1$$

It is customary to choose $R_f(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-1}$ since a constant disturbance then is eliminated, as was shown above. The controller is then said to have *integral action*, since the feedback part $S/R$ has the factor $1/(1-\mathrm{q}^{-1})$ which is a discretization of a continuous-time integrator (in discrete-time it should rather be called summator). It is usually not necessary to include a fixed factor in $S$. Suitable noise reduction can be achieved instead by a proper selection of $A_c$.

**Example: 4.2**

A sparc ignited (SI) engine of a car needs a control system that defines the timing of the sparc ignition with respect to the combustion cycle. After ignition the pressure increases in the cylinder and reaches a maximum peak value whereafter it decreases again. For an optimal performance trade-off between force and heat loss the pressure peak position (PPP) should be kept constant at a given crank angle. The dynamics can approximately be described by

$$y(k) = bu(k-1) + \text{disturbances}$$

where $k$ is the combustion cycle, $b = 0.65$, and $y$, $u$ are the PPP and the ignition time after top-end-position, respectively.

A controller with integral action is required in order to eliminate the slow varying disturbance. With usual formalism, the system is described as $Ay = Bu$, where $A = 1$ and $B = b\mathrm{q}^{-1}$. Integral action is obtained by the pre-chosen factor $R_f = 1 - \mathrm{q}^{-1}$. The closed-loop characteristic polynomial is chosen as $A_c = 1 - \lambda\mathrm{q}^{-1}$ and $R_1 = 1$ ($\deg R_1 = \deg B - 1 = 0$) and $S = s_0$ ($\deg S = \deg R_f - 1 = 0$). The polynomial equation $AR + BS = A_c$ is

$$(1-\mathrm{q}^{-1}) + b\mathrm{q}^{-1}s_0 = A_c = 1 - \lambda\mathrm{q}^{-1} \Rightarrow s_0 = (1-\lambda)/b$$

Introduce a reference $r = PPP_{ref}$ and choose $T$ to give asymptotic tracking $y(\infty) = r(\infty)$. Since $y(\infty) = \frac{BT}{A_c}(1)r(\infty)$ if follows that

$$T = \frac{A_c}{B}(1) = S(1) = s_0 \quad \rightarrow u = \frac{s_0}{1-\mathrm{q}^{-1}}e \quad \text{integral controller}$$

The result from pole placement at $\lambda = 0.5$ is shown in Fig. 4.1. Notice that the more commonly encountered PI controller is unnecessarily complex for this

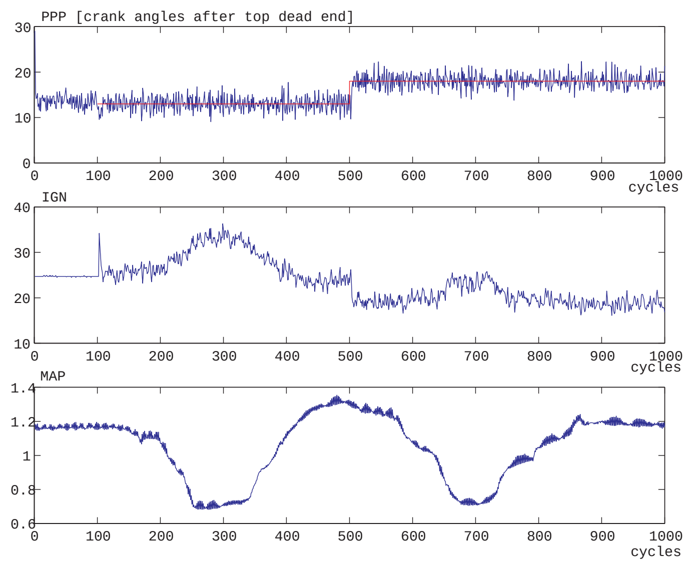



Figure 4.1: Real data from a SAAB 9000 turbo. The I-controller in Example 4.2 is used. $y =$ PPP (pressure peak position) is controlled and a step response is shown after 500 cycles. IGN is the ignition time in degrees before top-end-position. The control signal in the example is therefore $u = -IGN$. MAP (manifold pressure) is the load.

example and would probably perform worse due to the augmented dynamics of the closed loop (see Exercise 2).

□

The following example shows that a discrete-time designed controller can outperform a discretized PD controller, making the closed-loop response linear despite that the plant is strongly nonlinear. The effect of the pre-chosen factor $S_f = 1 + \mathrm{q}^{-1}$ to reduce noise feedback is also illustrated.

**Example: 4.3**

A magnetic suspension system is shown in Fig. 4.2. The dynamics of the suspending ball is

$$m\ddot{x} = F_g - F_m \qquad \begin{cases} F_g = mg & \text{gravitational force} \\ F_m = -\frac{1}{2}\frac{dL}{dx}i^2 & \text{magnetic force} \end{cases}$$

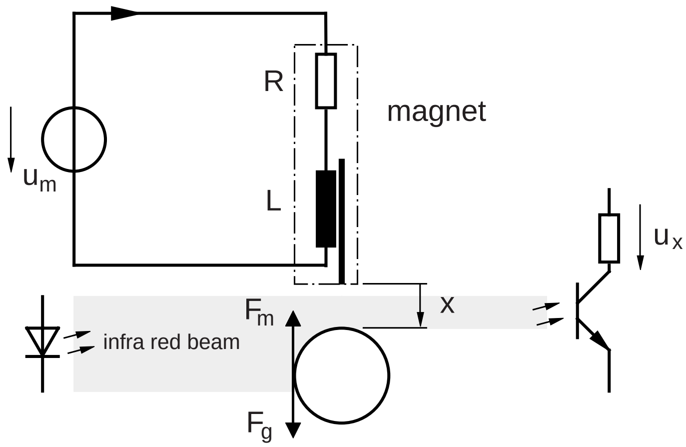



Figure 4.2: A magnetic suspension system.

Since $\frac{dL}{dx}$ depends on $x$, linearization of the dynamics gives the transfer function

$$G(s) = \frac{b}{s^2 - a}$$

Zero-order-hold sampling (see Chapter 5) results in the discrete-time system

$$H(q^{-1}) = \gamma \frac{q^{-1} + q^{-2}}{1 - \beta q^{-1} + q^{-2}}, \qquad \beta > 0$$

The parameters are estimated to $\beta = 2.0203$ and $\gamma = 0.9217$. A discrete-time PD controller gives the performance shown in Fig. 4.3. The response is highly nonlinear. A pole-placement design is now made on the discrete-time model. Integral action is introduced by choosing $R_f = 1 - q^{-1}$. Let $\deg A_c = \deg A' + \deg B - 1 = 4$, $\deg R_1 = \deg B - 1 = 1$, $R = R_1 R_f$ and $\deg S = \deg A' - 1 = 2$. The characteristic polynomial is chosen $A_c = \prod_{i=1}^{4}(1 - \lambda_i q^{-1})$, $\lambda_i = 0.95, 0.54, 0.33, 0.21$. The resulting response is shown in Fig. 4.4. Notice the linear behavior. The response is close to the response of the linear model, giving a standard deviation of the unmodeled response to $10\mu m$. However, the control signal is rather noisy. In order to suppress the feedback of measurement noise to the actuator, the pre-chosen factor $S_f = 1 + q^{-1}$ is chosen. The resulting response is shown in Fig. 4.5. The unmodeled response is here larger but the control signal much calmer. The trade-off between the standard deviation of the unmodeled response and noisy control signal is illustrated in Table 4.1.

□

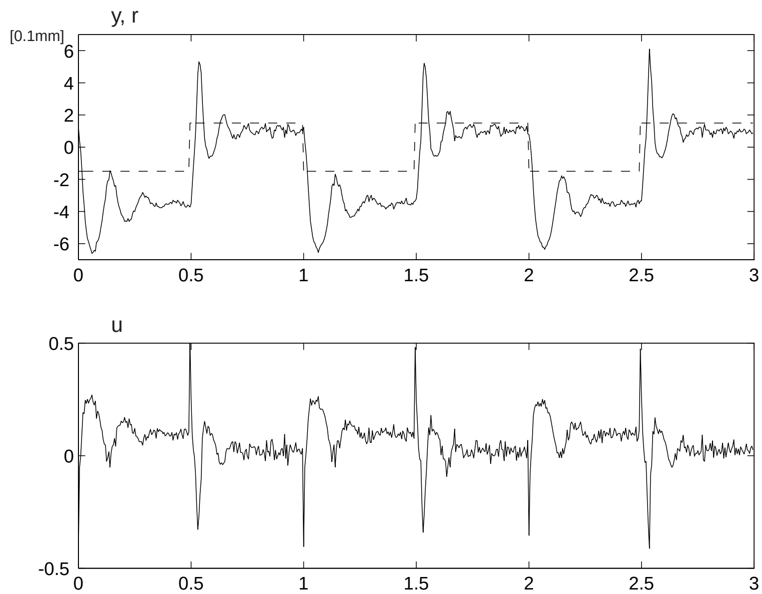



Figure 4.3: Discretized PD control of the magnetic suspension system.

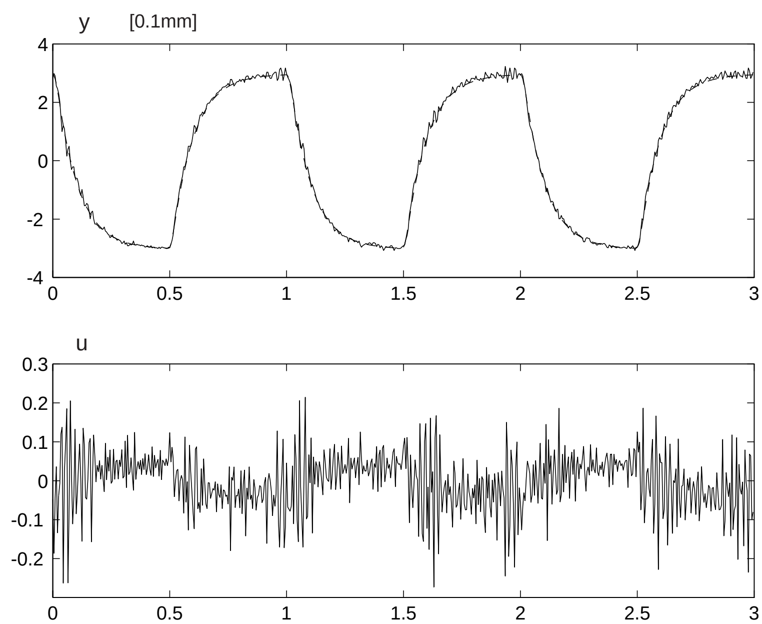



Figure 4.4: Pole-placement control of the magnetic suspension system using the pre-chosen factor $R_f = 1 - \mathrm{q}^{-1}$ for integral action.

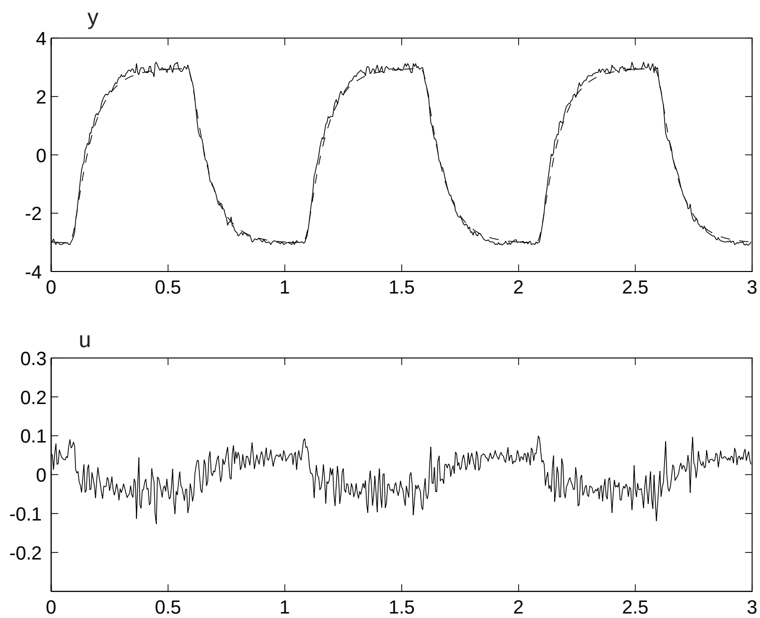



Figure 4.5: Pole-placement control of the magnetic suspension system using the pre-chosen factors $R_f = 1 - \mathrm{q}^{-1}$ for integral action and $S_f = 1 + \mathrm{q}^{-1}$ for less noise feedback.

| $S_f$ | $STD(e_u)$ [$\mu$m] | $\max \lvert \Delta u \rvert$ [V] |
|---|---|---|
| 1 | 10 | 0.45 |
| $1 + \mathrm{q}^{-1}$ | 15 | 0.12 |

Table 4.1: Model mismatch versus noise feedback in the control signal, influenced by the pre-chosen factor $S_f$.

## 4.2 Model matching

The design can be regarded as a model-matching problem. A desired performance can be specified by the introduction of a reference model:

$$y_r(k) = H_r^R(\mathrm{q}^{-1})r(k)$$

The output of the closed-loop system is described by

$$y(k) = H_r(\mathrm{q}^{-1})r(k) + H_\varphi(\mathrm{q}^{-1})\varphi(k) = \frac{BT}{A_c}r(k) + \frac{CR}{A_c}\varphi(k)$$

The model-matching objective is simply that

$$H_r = H_r^R$$

Thus, $y(k) = y_r(k), \forall k$ when $H_\varphi(\mathrm{q}^{-1})\varphi(k) = 0$. Notice that this does not imply that $A_c$ is stable since unstable pole-zero cancellations may occur (making $u \to \infty, k \to \infty$). Therefore, any reference model $H_r^R$ cannot be tolerable. It must be chosen in such a way that no unstable pole-zero cancellation occurs. The reference model is then said to be *compliant*. In order to take this constraint into account, factorize the numerator polynomial as $B = B^+B^-$, such that $B^+$ is stable (i.e. $|z_i| < 1 \forall z_i : B^+(z_i^{-1}) = 0$), monic (i.e. $B^+(0) = 1$) and $B^-$ unstable. The reference model is then compliant if

$$H_r^R = \frac{B^-B^R}{A_c^R}$$

where $A_c^R$ and $B^R$ is a stable and arbitrary polynomial, respectively. The model-matching requirement $H_r = H_r^R$ then gives

$$H_r = \frac{B^+B^-T}{A_c} = \frac{B^-B^R}{A_c^R} = H_r^R$$

resulting in the conditions

$$T = B^R A_o$$
$$A_c = A_c^R B^+ A_o$$

Extra freedom is introduced by a stable, monic polynomial $A_o$ which is canceled in $H_r$. Notice, though, that $A_o$ is not canceled in $H_\varphi$ and can therefore be used for an independent assessment of the disturbance response.

## Special cases

When $B$ does not have any unstable zeros, i.e. $B^- = b_d \mathrm{q}^{-\mathrm{d}}$, it is possible to cancel all zeros. We can then choose $B^R = 1/b_d$ and $A_c^R = 1$ resulting in the reference model $H^R = \mathrm{q}^{-\mathrm{d}}$. This is the fastest response one can demand from a system. Suppose further that there is no extra time-delay in the system, i.e. $d = 1$. In that case, the polynomial equation has an explicit solution in polynomial form

$$\begin{cases} R = R_f B^+ \\ S = (A_o - AR_f)/(b_1 \mathrm{q}^{-1}) \\ T = A_o/b_1 \end{cases}$$

which results in

$$\begin{array}{l} H_r = \mathrm{q}^{-1} \\ H_\varphi = \frac{R_f C}{A_o} \end{array}$$

It is instructive to compare with the corresponding internal-model controller (IMC). The IMC strategy is to cancel all process dynamics, i.e. not only the zeros, but also the poles. The model $G_u = B/A$ is therefore inverted approximately by the choice $G_r = \mathrm{q}^{-1}A/B$ resulting in the feedback controller $G_e = G_r/(1 - G_r G_u) = \mathrm{q}^{-1}A/(B(1 - \mathrm{q}^{-1})) = S/R$. Thus, in RST-form the IMC controller becomes

$$\begin{cases} R = \frac{B}{\mathrm{q}^{-1}}(1 - \mathrm{q}^{-1}) \\ S = A \\ T = A \end{cases}$$

Notice, that the IMC always has integral action. It is imposed by the particular structure. To get integral action in the pole-placement controller we need to choose $R_f = 1 - \mathrm{q}^{-1}$. The two designs then result in

$$\begin{array}{ll} y(k) = \mathrm{q}^{-1} r(k) + \frac{(1-\mathrm{q}^{-1})C}{A_o}\varphi(k) & \text{with pole placement} \\ y(k) = \mathrm{q}^{-1} r(k) + \frac{(1-\mathrm{q}^{-1})C}{A}\varphi(k) & \text{with IMC} \end{array}$$

Notice the important difference: $A_o$ is a stable polynomial freely chosen by the designer while $A$ is the denominator polynomial of the plant. Therefore, IMC cannot be used like this if the system is unstable. It is also inappropriate for stable systems with badly damped poles. This is illustrated below.

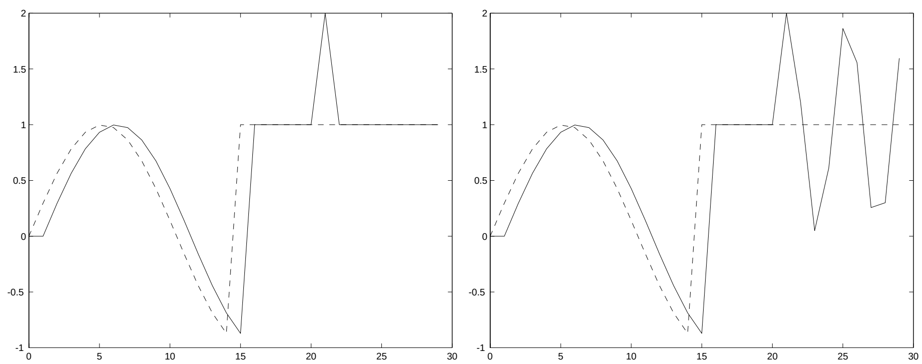



Figure 4.6: Pole-placement control (left) and internal-model control (right). The reference responses are the same, but not the disturbance responses.

**Example: 4.4**

Consider the system

$$y(k) = \frac{\mathrm{q}^{-1}}{A}[u(k) + \varphi(k)]$$

where $A = (1 - \lambda \mathrm{q}^{-1})(1 - \bar{\lambda}\mathrm{q}^{-1})$, $\lambda = 0.1 + \mathrm{i}0.99$. Thus, $A$ is stable but with very badly damped oscillatory poles. However, there are no zeros to worry about. The disturbance $\varphi$ enters at the process input, modeled by $C = B = \mathrm{q}^{-1}$. For the fastest disturbance rejection we can choose $A_o = 1$. The two controllers then give

$$\begin{aligned} y(k) &= r(k-1) + (1 - \mathrm{q}^{-1})\varphi(k-1) && \text{with pole placement} \\ y(k) &= r(k-1) + \tfrac{(1-\mathrm{q}^{-1})}{A}\varphi(k-1) && \text{with IMC} \end{aligned}$$

If the disturbance is a step, consider e.g. $\varphi(k) = 1$, $t \geq 20$, then its influence on $y$ is eliminated in finite time for the inverse controller. In fact, $y(22) = r(21) + \varphi(21) - \varphi(20) = r(21)$. Conversely, for IMC the badly damped poles are excited and shown in the response for a long time, see Fig. 4.6. □

The above example showed that it can be dangerous to cancel badly damped poles. In exercise 1 it is shown that cancellation of 'badly damped zeros' should be avoided as well.

## 4.3 Example: Servo control designs

A servo control system will now show how different design choices influence the closed-loop behavior. The system consists of an inner loop which is an analog

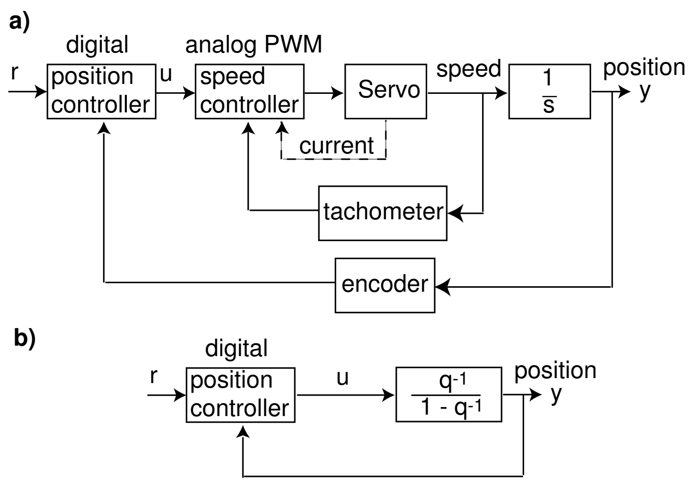



Figure 4.7: A servo control system in a cascade control structure. The inner loop is a pulse-width modulated proportional controller and the outer loop is the digital control problem.

pulse-width modulated proportional control system, and an outer loop, which is the digital control system under investigation, see Fig. 4.7. The dynamics, from the digital control point of view, is simply an integrator (summator)

$$y(k) = y(k-1) + u(k-1)$$

where the gain has been scaled to one for simplicity. Thus, in polynomial form the system is

$$A(\mathrm{q}^{-1})y(k) = B(\mathrm{q}^{-1})u(k), \quad A(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-1}, \quad B(\mathrm{q}^{-1}) = \mathrm{q}^{-1}$$

Due to an erroneous calibration of the power supply, there is a constant drift of the servo even though the control input is zero. In order to eliminate bias, the controller needs integral action, i.e. $R_f = 1 - \mathrm{q}^{-1}$. Therefore, the number of closed-loop poles to be chosen is $\deg A_c = \deg(AR_f) + \deg B - 1 = 2$, and the controller polynomials satisfy $\deg R_1 = \deg B - 1 = 0$, $\deg S = \deg(AR_f) - 1 = 1$ where $R = R_f R_1 = R_f$.

The different design examples below will illustrate various aspects of pole placement design. First, a one-degree-of-freedom structure is chosen to show the influence from a zero in the reference response dynamics. Then, the two-degree-of-freedom structure is chosen which allows to avoid the introduction of the zero.

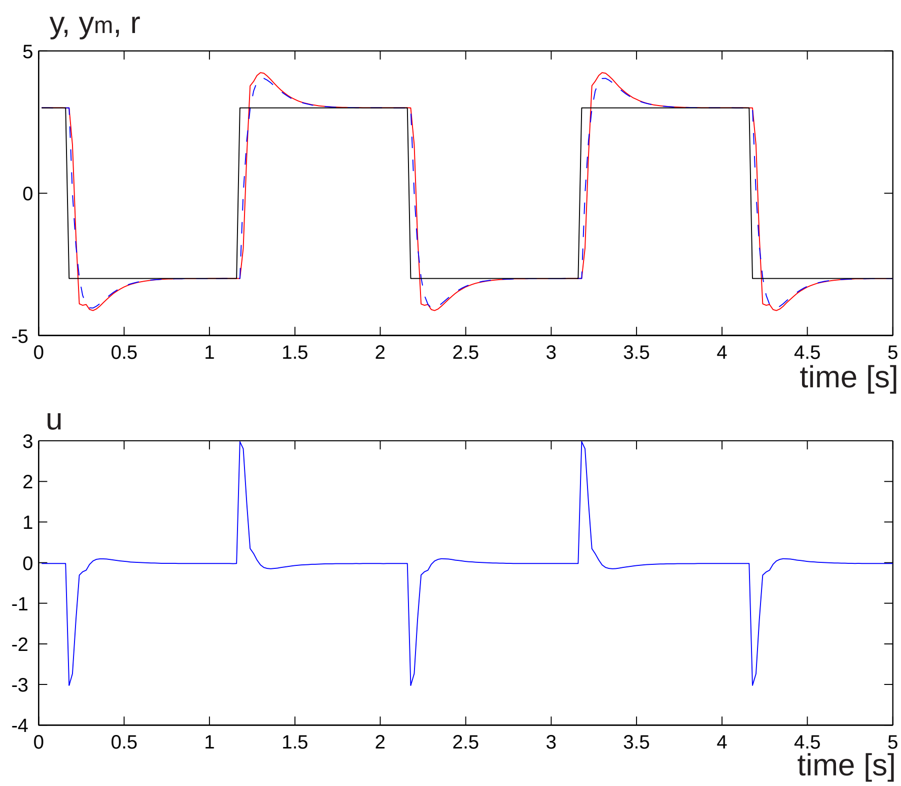



Figure 4.8: Example 4.5 showing the servo system response when $T = S$. Above: actual response $y$ (solid), modeled response $y_m$ (dashed) and reference $r$. Below: control signal $u$.

The very important aspect of noise sensitivity is then illustrated in the last two examples where one design is noise sensitive and the other is not.

**Example: 4.5**

The two poles are chosen as $\lambda_1 = 0.7$ and $\lambda_2 = 0.8$. Let us try a one-degree-of freedom controller by choosing $T = S$. This results in an overshoot in the step responses, see Fig. 4.8. This is because $T$ is of first order and has a zero close to one, resulting in almost a differentiation of the reference. Thus, despite the closed-loop poles are chosen to be 'non-oscillatory', well-damped on the real axis, the step response gets an overshoot because of the zero introduced in $T$. □

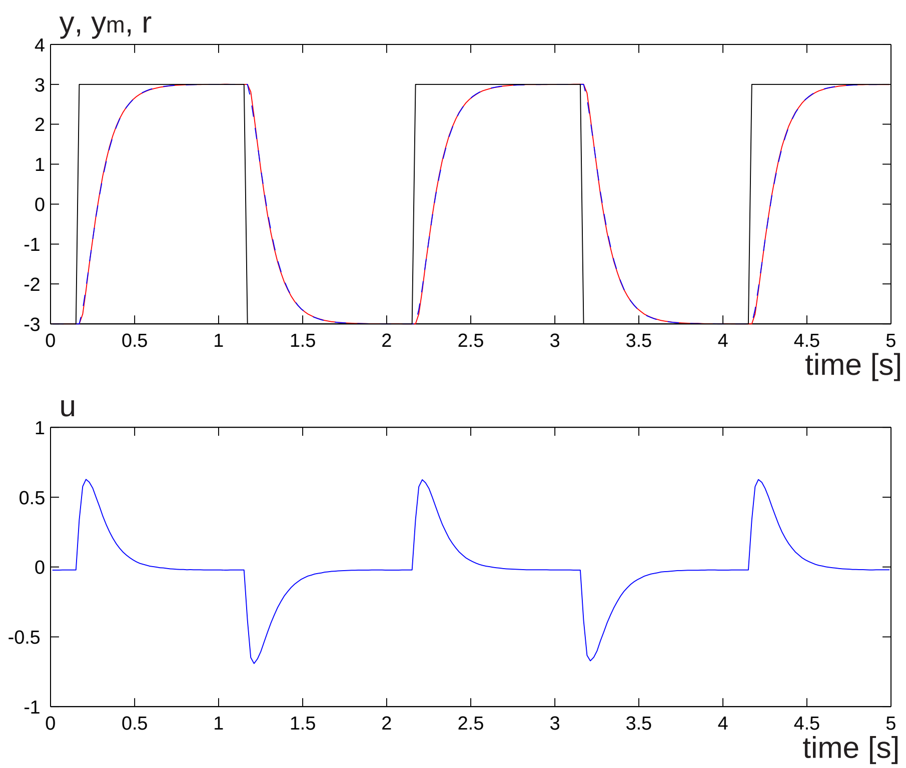



Figure 4.9: Example 4.6 showing the servo system response when $T = S(1)$. Above: actual response $y$ (solid), modeled response $y_m$ (dashed) and reference $r$. Below: control signal $u$.

**Example: 4.6**

Let us modify the previous design by selecting $T = S(1)$. This results in a step response without overshoot as expected from the pole placement, see Fig. 4.9. □

**Example: 4.7**

When removing the zero of $T$ in the above example, the response becomes much slower and corresponds directly to the rather 'slow' pole placement. If a faster response is required, the poles should be moved closer to the origin. Let us try $A_c = 1 - 0.7\mathrm{q}^{-1}$ which would result in a faster response than the previous design since the pole at $0.8$ has been moved to zero. The response is shown in Fig. 4.10 and is faster as expected. However, it is also oscillatory which is not what we

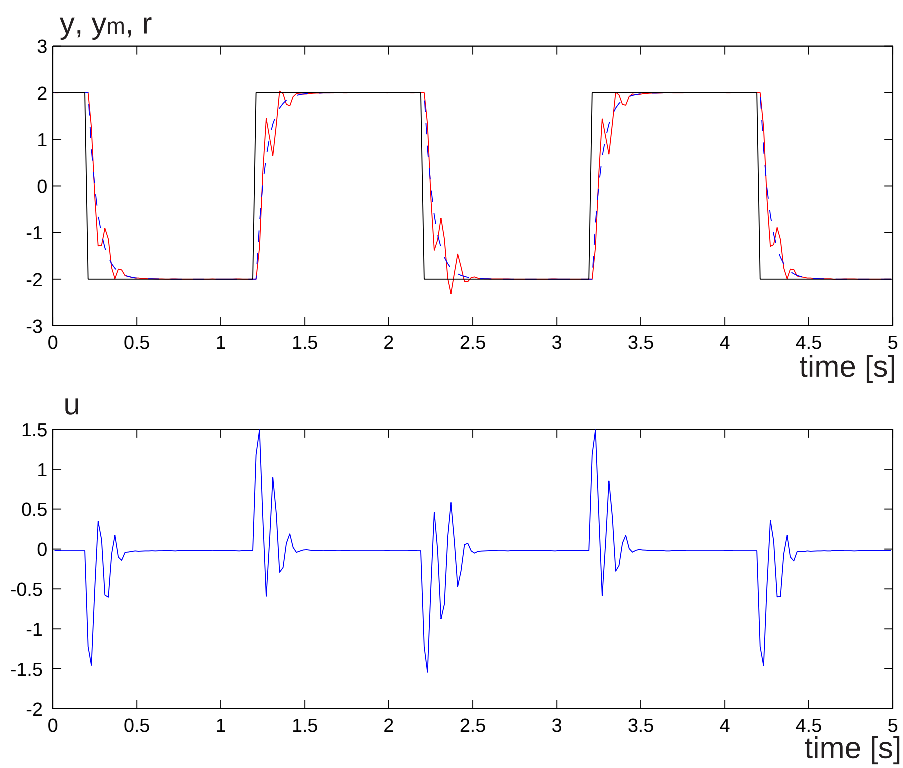



Figure 4.10: Example 4.7 showing the servo system response when $S_u = |\frac{AS}{A_c}(-1)| = 2.7$. Above: actual response $y$ (solid), modeled response $y_m$ (dashed) and reference $r$. Below: control signal $u$.

designed for. The discrepancy between the modeled and the actual responses is because the design is sensitive and the controller excites higher order unmodeled dynamics. This design shows the danger by just focusing on the reference response dynamics and not consider the disturbance response in the design as well. The disturbance response from additive output noise to the control signal is governed by

$$u(k) = -\frac{AS}{A_c}(\mathrm{q}^{-1})\varphi(k)$$

A measure of noise sensitivity of the control signal can be defined as the evaluation of the above amplitude response at the highest frequency $\omega = \pi$. This gives

$$S_u = |\frac{AS}{A_c}(-1)|$$

In this example, $S_u = 2.7$. A good design should try to make $S_u$ as small as possible. □

**Example: 4.8**

Now reformulate the previous design such that the designed fast reference response is obtained. This can be done by selecting another, less noise sensitive, solution to the model matching problem. Choose, $A_c = (1 - 0.7\mathrm{q}^{-1})(1 - 0.8\mathrm{q}^{-1})$ as earlier and cancel the slow pole at $0.8$ by $T$, i.e. $T = (1 - 0.8\mathrm{q}^{-1})S(1)/0.2$. Notice that we make $T(1) = \frac{A_c}{B}(1) = S(1)$ to make the steady-state gain one. We now have the same $\frac{BT}{A_c}$ as in the previous design since the factor $(1 - 0.8\mathrm{q}^{-1})$ is canceled between $T$ and $A_c$. However, this factor is not canceled in the disturbance responses. This reduces considerably the excitation of unmodeled dynamics. The noise sensitivity defined above is here $S_u = 0.6$. The resulting response is shown in Fig. 4.11. Notice the close fit between the modeled and the actual response. We got what we designed for! □

## 4.4 Example: Damping of a pendulum

A hanging pendulum connected to a moving trailer will now be studied, see Fig. 4.12. The purpose with this example is to show that although there are more than one output to control, the developed pole placement approach can still be used. The two output variables are the trailer position $x$ and the pendulum angle $y$. The problem is turned into the familiar framework by cascade control, see Fig. 4.13. The advantage with this control structure is that the overall problem can be split into smaller parts that are solved successively. The inner loop is first designed, thereafter the outer one.

### The inner loop—control of the trailer

The inner loop control problem consists of the trailer that is controlled by a servo (the same kind as in the previous example). The dominating dynamics is therefore an integrator. But due to a flexible transmission (a rubber band) between the servo and the trailer, there are also higher order dynamics present. The servo-to-trailer dynamics $u \to x$ are identified with the structure $A_x x = B_x u$ where

$$\begin{cases} A_x = 1 + a_1\mathrm{q}^{-1} + a_2\mathrm{q}^{-2} + a_3\mathrm{q}^{-3} \\ B_x = b_1\mathrm{q}^{-1} + b_2\mathrm{q}^{-2} + b_3\mathrm{q}^{-3} \end{cases}$$

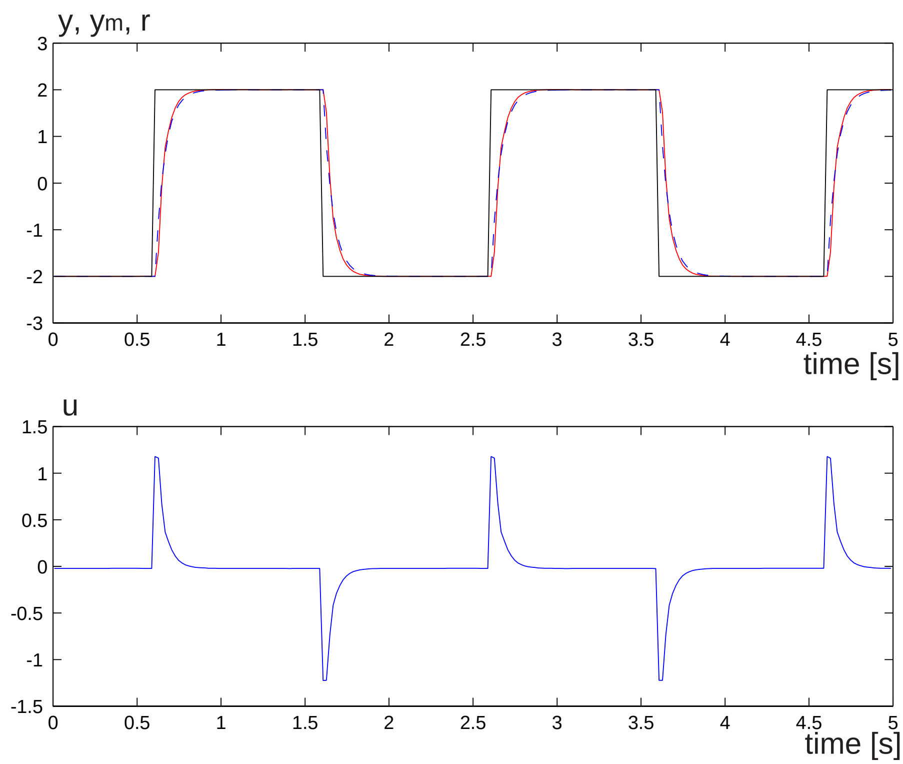



Figure 4.11: Example 4.8 showing the servo system response when $S_u = |\frac{AS}{A_c}(-1)| = 0.6$. Above: actual response $y$ (solid), modeled response $y_m$ (dashed) and reference $r$. Below: control signal $u$.

For simplicity a P-controller is chosen, i.e. $u = Ke$, $e = v - x$. The closed inner loop then becomes $A_1 x = B_1 v$ where

$$\begin{cases} A_1 = A_x + KB_x \\ B_1 = KB_x \end{cases}$$

## The outer loop—control of the pendulum

The pendulum dynamics $x \to y$ have the expected structure $A_y y = B_y x$ with

$$\begin{cases} A_y = (1 - \lambda q^{-1})(1 - \lambda^* q^{-1}) \\ B_y = b_0(1 - q^{-1})^2 \end{cases}$$

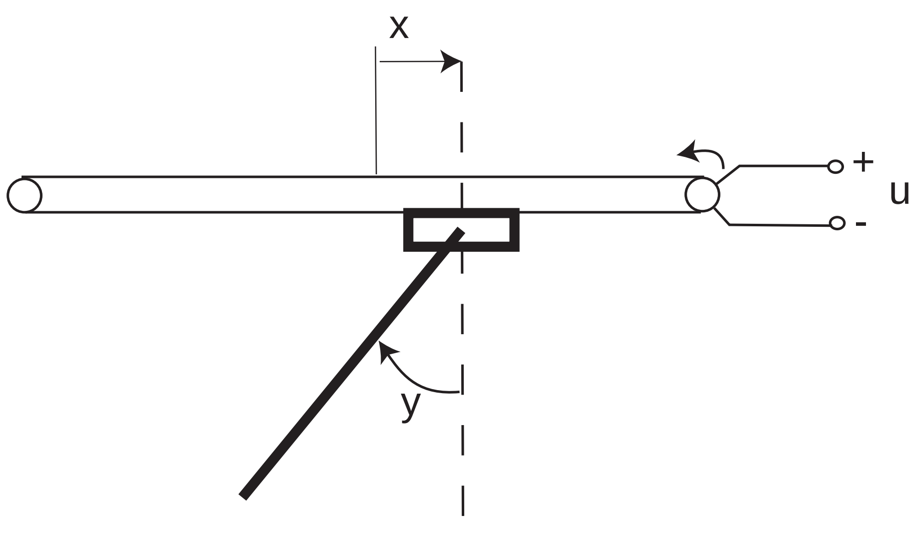



Figure 4.12: A hanging pendulum connected to a moving trailer.

where the complex poles $\lambda$, $\lambda^* = \mathrm{conj}[\lambda]$ are close to the unit circle, giving an oscillatory badly damped response. The double zero at $1$ comes from the fact that the position is differentiated twice to become acceleration which is proportional to the force on the trailer. The parameters are estimated from experiments. The outer loop 'plant' model now is

$$\begin{cases} A_2 = A_1 A_y & \begin{cases} A_1 \text{ well damped} \\ A_y \text{ badly damped} \end{cases} \\ B_2 = B_1 B_y \end{cases}$$

Keep $A_1$ in $A_c$. According to the polynomial equation it then must be included in $S$ as well.

$$\begin{cases} A_c = A_1 A_{c1} \\ S = A_1 S_1 \end{cases}$$

After cancellation of the common factor $A_1$, the polynomial equation becomes

$$A_y R + S_1 B_2 = A_{c1}$$

$$\begin{cases} \deg A_{c1} \leq \deg A_y + \deg B_2 - 1 = 6 \\ \deg R = \deg B_2 - 1 = 4 \\ \deg S_1 = \deg A_y - 1 = 1 \end{cases}$$

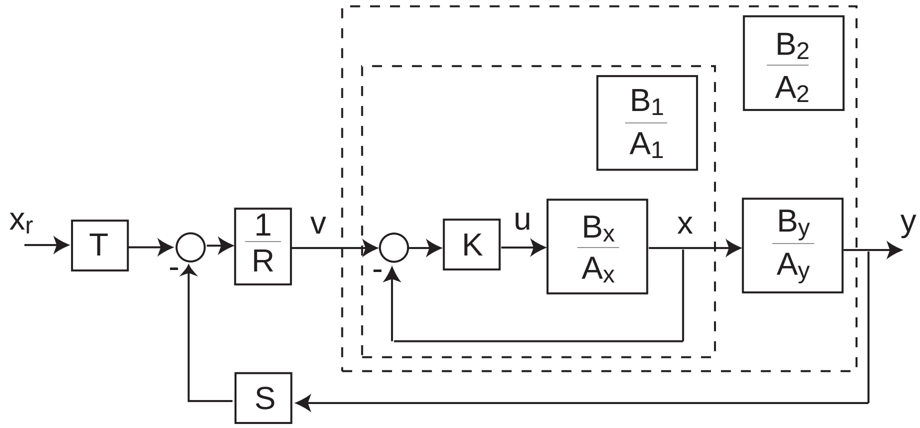



Figure 4.13: Cascade control of the pendulum-chart system.

Well damped poles are chosen as

$$A_{c1} = \prod_{k=1}^{6}(1 - \lambda_k q^{-1})$$
$$\lambda_k = 0.85, 0.8, 0.75, 0.7, 0.65, 0.6$$

The resulting well-damped response is shown in Fig. 4.14, where also the undamped (open-loop) response is shown for comparison.

## Introducing a reference position

A reference $r$ to the position $x$ can be introduced by selecting $T$ scalar such that the steady-state gain is 1. The resulting response is shown in Fig. 4.15. Notice that the pendulum angle $y$ is kept well-damped despite the rather large move of the trailer (position $x$).

$$x(\infty) = v(\infty) = \frac{A_2 T}{A_c}(1) r(\infty), \quad \rightarrow T = \frac{A_c}{A_2}(1) = R(1)$$

# Exercises

A DC motor with transfer function

$$G_{cont}(s) = \frac{4}{s(s+2)}$$

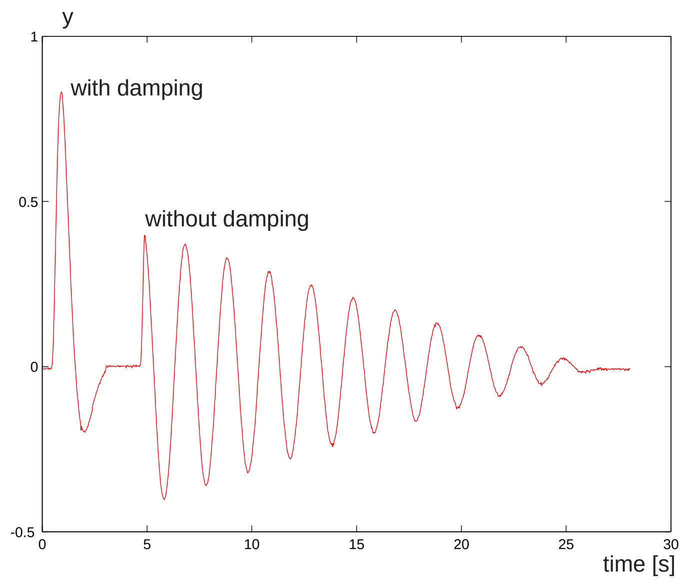



Figure 4.14: Active damping of the pendulum by cascade control. The undamped response is also shown for comparison.

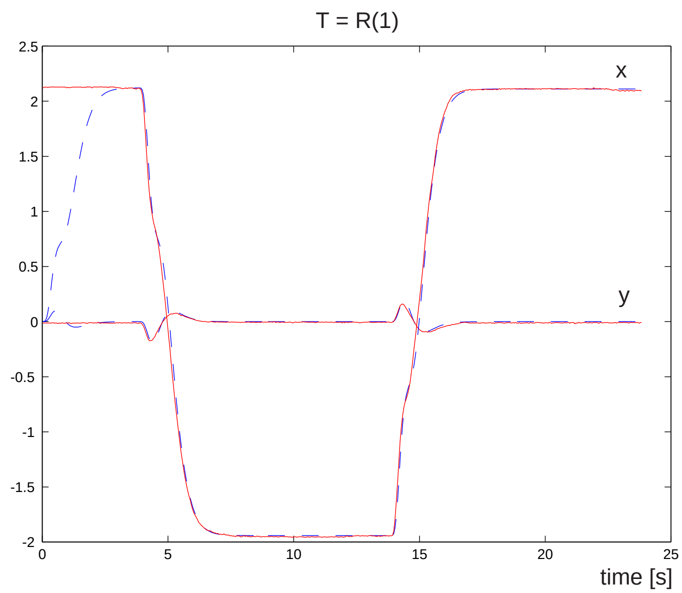



Figure 4.15: Step responses of the trailer position $x$ during active damping of the pendulum angle $y$. The modeled responses are dashed.

is sampled with a period of $0.025$ s, resulting in a discrete-time system

$$G_{disc}(\mathrm{q}^{-1}) = \frac{10^{-3}(1.23\mathrm{q}^{-1} + 1.21\mathrm{q}^{-2})}{1 - 1.95\mathrm{q}^{-1} + 0.95\mathrm{q}^{-2}} = \frac{1.23 \cdot 10^{-3}\mathrm{q}^{-1}(1 + 0.98\mathrm{q}^{-1})}{(1 - \mathrm{q}^{-1})(1 - 0.95\mathrm{q}^{-1})}$$

**1a)** Calculate a controller that makes the closed-loop system

$$H(\mathrm{q}^{-1}) = \frac{0.2\mathrm{q}^{-1}}{1 - 0.8\mathrm{q}^{-1}}$$

**1b)** Calculate $R$ and $S$ that makes the closed-loop characteristic polynomial

$$A_c(\mathrm{q}^{-1}) = 1 - 0.8\mathrm{q}^{-1}$$

and choose $T$ (scalar) such that the closed-loop system has unit steady-state gain.

**1c)** Suppose $\varphi(k)$ adds to $y(k)$ as measurements noise. The transfer operator from $\varphi$ to the control $u$ is then

$$u(k) = -\frac{SA}{A_c}\varphi(k)$$

Evaluate the noise amplification to the actuator as

$$|\frac{S(-1)A(-1)}{A_c(-1)}|$$

Choose $A_c = (1 - 0.95\mathrm{q}^{-1})(1 - 0.93\mathrm{q}^{-1})(1 - 0.9\mathrm{q}^{-1})$ and calculate the noise gain of the control system as above. Compare to the designs in **1a,b)**.

**2** Suppose PPP in Example 4.2 is controlled by the discrete-time PI controller

$$u(k) = K[1 + \frac{1}{T_i(1 - \mathrm{q}^{-1})}]e(k)$$

where $e = r - y$. What is $A_c = \prod_i^n (1 - \lambda_i \mathrm{q}^{-1})$? Is it possible (with $K, T_i > 0$) to make all $\lambda_i > 0$?

**Solutions**

**1a.** $A_c = (1-0.8\mathrm{q}^{-1})(1+0.98\mathrm{q}^{-1})$. From this we directly conclude that $R = 1+0.98\mathrm{q}^{-1}$ and $T = A_c(1)/B(1) = (1-0.8)/(1.23\cdot 10^{-3}) = 163$. Divide $AR+BS = A_c$ with $1+0.98\mathrm{q}^{-1}$ since it is a factor in all terms. Thus,

$$(1-\mathrm{q}^{-1})(1-0.95\mathrm{q}^{-1}) + 1.23\cdot 10^{-3}(s_0+s_1\mathrm{q}^{-1}) = 1-0.8\mathrm{q}^{-1}$$

Evaluate at $q=1$ and $q=0.95$ to get two equations including $s_0$ and $s_1$. These are easily calculated, resulting in $S = 935-772\mathrm{q}^{-1}$.

**1b.** $A_c = 1-0.8\mathrm{q}^{-1}$. Therefore $T = A_c(1)/B(1) = 81.97$. Evaluate the polynomial equation $AR+BS = A_c$ at the zero $-1/0.98$ of $B$ to get

$$A(-1/0.98)R(-1/0.98) = A_c(-1/0.98) \Rightarrow r_1 = -0.98[\frac{A_c(-1/0.98)}{A(-1/0.98)} - 1]$$

Similarly, evaluate at the zeros $1$ and $1/0.95$ of $A$ to get two independent equations from which $s_0$ and $s_1$ can be calculated, giving $S = 501-419\mathrm{q}^{-1}$.

**1c.** Denote $A_1 = 1-0.95\mathrm{q}^{-1}$. Then since $A = (1-\mathrm{q}^{-1})A_1$ and $A_c = A_{c1}A_1$ it follows that $S = S_1A_1$. Cancel the common factor $A_1$ from the polynomial equation to get $(1-\mathrm{q}^{-1})R + BS_1 = A_{c1}$. Since $\deg S_1 = \deg(1-\mathrm{q}^{-1})-1 = 0$ we get $S = s_0$ and the scalar can directly be calculated by evaluating the equation at $q=1$. This gives $S = s_0 = A_{c1}(1)/B(1) = 2.87$. Then $S(-1)A(-1)/A_c(-1) \approx 3$. Corresponding values for 1a and 1b are $\approx 1.8\cdot 10^5$ and $2000$. In both of these cases the higher response speed is to the cost of an enormous noise sensitivity. Consequently, 1a and 1b are rather stupid designs that make no sense.

**2.** The PI controller can be written as $u = \frac{S}{R}e$ with $R = 1-\mathrm{q}^{-1}$ and $S = s_0+s_1\mathrm{q}^{-1}$, where $s_0 = K(1+1/T_i)$ and $s_1 = -K$.

$$AR+BS = 1-\mathrm{q}^{-1} + b\mathrm{q}^{-1}(s_0+s_1\mathrm{q}^{-1}) = 1+(-1+bs_0)\mathrm{q}^{-1} + bs_1\mathrm{q}^{-2}$$

Compare with $A_c = (1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1}) = 1-(\lambda_1+\lambda_2)\mathrm{q}^{-1}+\lambda_1\lambda_2\mathrm{q}^{-2}$. Thus, $\lambda_1\lambda_2 = bs_1 = -0.65K < 0$. Not possible to choose both $\lambda_1$ and $\lambda_2$ positive with $K > 0$.

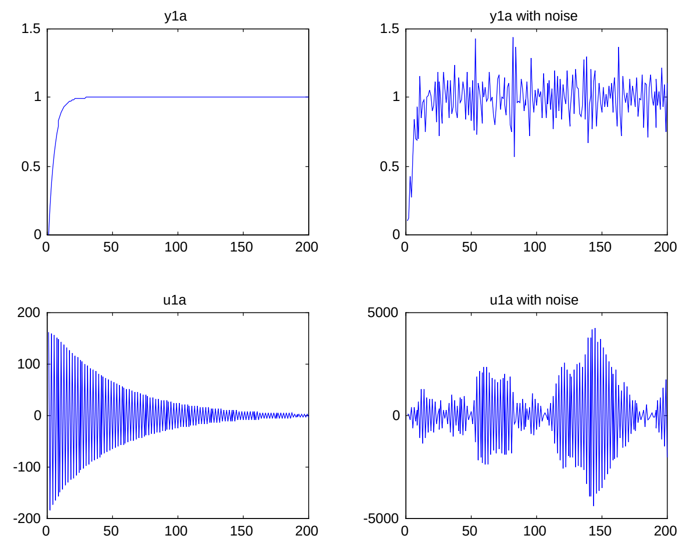



Figure 4.16: Simulation of Exercise 1a. Notice the 'rippling' in the control signal. This is caused by the cancellation of a zero close to $-1$.

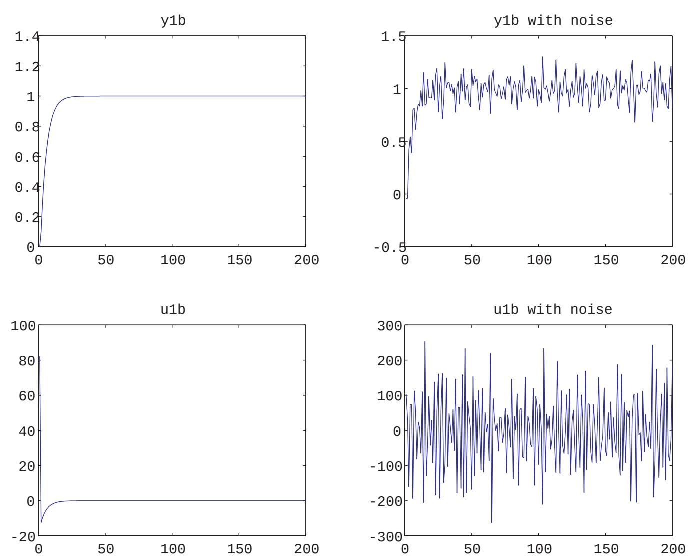



Figure 4.17: Simulation of Exercise 1b. Compared to Exercise 1a, the output response $y$ is the same, but without the rippling in the control signal. However, the noise sensitivity is still enormous.

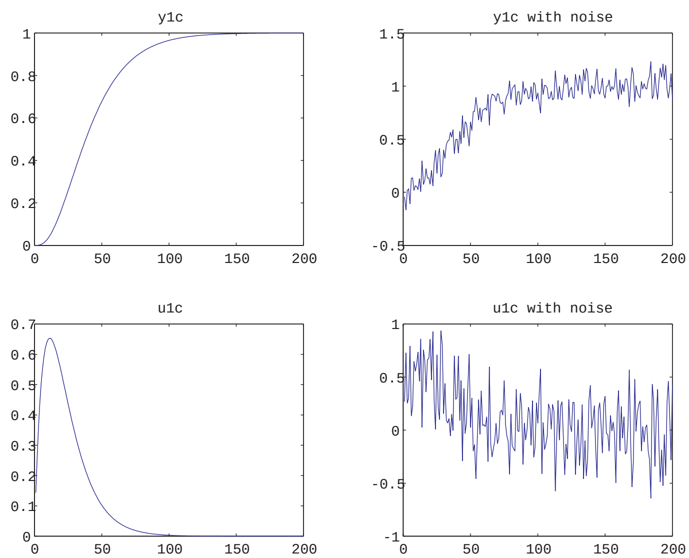



Figure 4.18: Simulation of Exercise 1c. Slower response than in 1a-b, but much less control effort, giving quite an improvement regarding noise sensitivity.

# Chapter 5

# Sampling

In many cases, the discrete-time model represents a continuous-time model at equidistant sampling instants. How to derive the discrete-time model from knowledge of the continuous-time one is described in this chapter. The discrete-time model depends both on the sampling period and the hold device that specifies the control signal between the sampling instants. Usually, a zero-order-hold device is used that maintains the continuous-time control signal at the discrete-time value during the sampling period. This gives a staircase-like control signal.

The chapter is organized as follows. First, sampling of signals and occurrence of frequency folding or *aliasing* are illustrated. Then, sampling of systems with zero-order-hold inputs is described. Various sampling formula are given. Finally, the choice of sampling frequency is discussed and a rule of thumb is illustrated.

## 5.1 Sampling of signals

Consider the continuous-time signal

$$y(t) = sin(\omega t)$$

sampled at time instances $t = kh$, $k = 1, 2, \ldots$, where $h$ is the sampling period. The sampling frequency is defined as $f_s = 1/h$, or, in radians per time, $\omega_s = 2\pi/h$. The sampled signal $y(kh)$ can not be distinguished from the signal

$$z(t) = sin(\omega_a t)$$

sampled at the same time instants $t = kh$ and with $\omega_a$ satisfying

$$\omega_a = \omega + n\omega_s, n \in \{\pm 1, \pm 2, \ldots\}$$

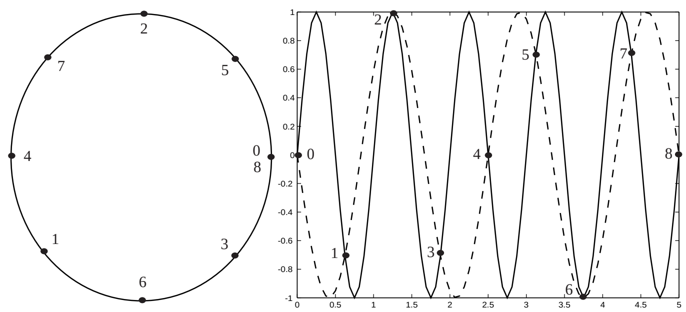



Figure 5.1: Sampling of $y(t) = sin(2\pi ft)$ and $z(t) = sin(2\pi f_a t)$ at $t = kh$, $k = 1, 2, \ldots$ where $h = 5/8$. The frequency $f = 1$ is larger than the Nyquist frequency $f_N = 1/(2h) = 4/5$ giving rise to the alias frequency $-f_a = f_N - (f - f_N) = 3/5$.

Thus, so called *alias* frequencies $\omega_a$ are introduced that were not present in the original continuous signal $y(t)$. Consequently, there is an ambiguity in the sampled representation. This is a problem if $\omega > \omega_s/2$ since then an alias $\omega_a$ is *folded* down below $\omega$ such that $|\omega_a| < |\omega|$ (the sign of the frequency is not relevant since it only results in a phase shift). The phenomenon is therefore also called frequency folding. The folding occurs around multiples of $\omega_N = \omega_s/2$, denoted the Nyquist frequency. This is seen from the reformulation

$$-\omega_a = n\omega_N - (\omega - n\omega_N)$$

**Example: 5.1**

Consider $y(t) = sin(\omega t)$ with $\omega = 2\pi f$ and $f = 1$. Sample with period $h = 5/8$. Then the sampling frequency is $f_s = 8/5$ and an alias frequency occurs at $f_a = f - f_s = -3/5$, i.e. $|f_a| = 3/5 < 1 = f$. The original continuous-time signal $y(t)$ (solid) and the alias signal $z(t)$ (dashed) are shown in Fig. 5.1. Notice that $y(kh) = z(kh)$.
□

## 5.2 Sampling of systems

We will now derive the relation between a continuous-time model $y(t) = G(\frac{d}{dt})u(t)$ with zero-order-hold (ZOH) inputs $u(t + kh) = u(kh)$, $0 \leq t < h$, $k = 0, 1, \ldots$ and the corresponding discrete-time model $y(kh) = H(\mathrm{q}^{-1})u(kh)$.

## A first order system

Consider the first-order system

$$\frac{dy(t)}{dt} = py(t) + u(t), \quad G(\frac{d}{dt}) = \frac{1}{\frac{d}{dt} - p}$$

Multiply with integrating factor to get

$$e^{-pt}(\frac{dy(t)}{dt} - py(t)) = \frac{d}{dt}(e^{-pt}y(t)) = e^{-pt}u(t)$$

Integrate from $0$ to $t$

$$e^{-pt}y(t) - y(0) = \int_0^t e^{-p\tau}u(\tau)d\tau = \int_0^t e^{-p(t-\tau)}u(t-\tau)d\tau$$

This gives

$$y(t) - e^{pt}y(0) = \int_0^t e^{p\tau}u(t-\tau)d\tau$$

Let $t = h$ and use zero-order-hold $u(h-t) = u(0)$, $0 < t \leq h$.

$$y(h) - e^{ph}y(0) = \begin{cases} \frac{1}{p}(e^{ph} - 1)u(0) & p \neq 0 \\ hu(0) & p = 0 \end{cases}$$

This corresponds to $k = 0$ while $y(kh)$, for any $k$, evolves as

$$y(kh+h) = \lambda y(kh) + cu(kh), \quad \begin{cases} \lambda = e^{ph} \\ c = \begin{cases} (\lambda - 1)/p & p \neq 0 \\ h & p = 0 \end{cases} \end{cases}$$

Thus, (with $s$ replacing $\frac{d}{dt}$ for brevity)

$$G(s) = \frac{1}{s-p} \quad \rightarrow \quad H(\mathrm{q}^{-1}) = \frac{c\mathrm{q}^{-1}}{1 - \lambda\mathrm{q}^{-1}}$$

## Higher order systems with distinct poles

Consider the $n$th-order system with distinct poles ($p_i \neq p_j$)

$$G(s) = \sum_{i=1}^{n} \frac{d_i}{s - p_i} = \frac{\sum_i d_i \prod_{j \neq i}(s - p_j)}{\prod_i (s - p_i)}$$

where we notice that $d_i = \lim_{s\to p_i}(s-p_i)G(s)$. Since sampling is a linear operation, it follows directly that the ZOH discrete-time system is

$$H(\mathrm{q}^{-1}) = \sum_{i=1}^{n} \mathrm{q}^{-1}\frac{c_i d_i}{1-\lambda_i \mathrm{q}^{-1}} = \mathrm{q}^{-1}\frac{\sum_i c_i d_i \prod_{j\neq i}(1-\lambda_j \mathrm{q}^{-1})}{\prod_i (1-\lambda_i \mathrm{q}^{-1})}$$

where similar as before we define

$$\begin{cases} \lambda_i = e^{p_i h} \\ c_i = \begin{cases} \frac{1}{p_i}(\lambda_i - 1), & p_i \neq 0 \\ h, & p_i = 0 \end{cases} \end{cases}$$

**Example: 5.2**

For $n=2$ and $p_1 \neq p_2$ it follows that

$$H(\mathrm{q}^{-1}) = \frac{b_1\mathrm{q}^{-1} + b_2\mathrm{q}^{-2}}{(1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1})}, \quad \begin{cases} b_1 = c_1 d_1 + c_2 d_2 \\ b_2 = -(c_1 d_1 \lambda_2 + c_2 d_2 \lambda_1) \end{cases}$$

□

## Multiple-pole systems

Consider the second order system

$$G(s) = \frac{1}{(s-p_1)(s-p_2)}$$

Let $\epsilon = p_2 - p_1$, then $d_1 = -d_2 = -1/\epsilon$. Clearly, the formula above cannot be used for a double-pole system since $d_1, d_2 \to \pm\infty$, when $\epsilon \to 0$. However, $b_1$ and $b_2$ are still bounded but need to be expressed differently. To see this, let first $p_1 = p$ and $p_2 = p_1 + \epsilon$ and express $b_1$ and $b_2$ as functions of $\epsilon$ using the formula above. Thereafter, let $\epsilon \to 0$, leaving bounded expressions.

First, consider the case $p \neq 0$. Use that $\lambda_2 = \lambda_1 \mathrm{e}^{\epsilon h} = \lambda_1 + \lambda_1(\mathrm{e}^{\epsilon h} - 1)$

$$\begin{aligned} b_1 &= c_1 d_1 + c_2 d_2 = \tfrac{\lambda_1 - 1}{p_1}\tfrac{-1}{\epsilon} + \tfrac{\lambda_2 - 1}{p_2}\tfrac{1}{\epsilon} = \tfrac{p_1(\lambda_2-1) - p_2(\lambda_1 - 1)}{\epsilon p_1 p_2} = \\ &= \tfrac{\epsilon + p_1\lambda_2 - p_2\lambda_1}{\epsilon p_1 p_2} = \tfrac{\epsilon - \lambda_1\epsilon + p_1\lambda_1(\mathrm{e}^{\epsilon h}-1)}{\epsilon p_1 p_2} = \tfrac{1-\lambda_1 + p_1\lambda_1((\mathrm{e}^{\epsilon h}-1))/\epsilon}{p_1 p_2} \end{aligned}$$

Now, use that $(\mathrm{e}^{\epsilon h} - 1))/\epsilon \to h$, $\epsilon \to 0$, to get $b_1 = (1 - \lambda + p\lambda h)/p^2$ where $\lambda = \mathrm{e}^{ph}$.

When $p = 0$ it follows that

$$\begin{aligned} b_1 &= c_1 d_1 + c_2 d_2 = h\tfrac{-1}{\epsilon} + \tfrac{\lambda_2 - 1}{\epsilon}\tfrac{1}{\epsilon} = \tfrac{1}{\epsilon}(-h + \tfrac{h\epsilon + (h\epsilon)^2/2 + \ldots}{\epsilon}) \\ &\to \tfrac{h^2}{2}, \quad \epsilon \to 0 \end{aligned}$$

Since $d_1 = d_2 = -1/\epsilon$, the above expression can be written $c_2 = c_1 + b_1\epsilon$. Using this in the formula for $b_2$ gives

$$\begin{aligned} b_2 &= -(c_1 d_1 \lambda_2 + c_2 d_2 \lambda_1) = -(c_1 \tfrac{-1}{\epsilon}\lambda_1 \mathrm{e}^{\epsilon h} + (c_1 + b_1\epsilon)\tfrac{1}{\epsilon}\lambda_1) \\ &= \lambda_1 (c_1 \tfrac{1 + \epsilon h + \ldots}{\epsilon} - \tfrac{c_1}{\epsilon} - b_1) \to \lambda(hc_1 - b_1) \quad \epsilon \to 0 \end{aligned}$$

To summarize, the $b$-parameters can be expressed as

$$\begin{cases} b_1 = \begin{cases} \frac{1 - \lambda + p\lambda h}{p^2} & p \neq 0 \\ \frac{h^2}{2} & p = 0 \end{cases} \\ b_2 = \lambda(hc_1 - b_1) \end{cases}$$

These expressions give a general formula for sampling of the double-pole system

$$G(s) = \frac{1}{(s-p)^2} \to H(\mathrm{q}^{-1}) = \frac{b_1 \mathrm{q}^{-1} + b_2 \mathrm{q}^{-2}}{(1 - \lambda \mathrm{q}^{-1})^2}$$

**Example: 5.3**

Sampling of a double integrator $G(s) = \frac{1}{s^2}$ results in $\lambda = 1$, $b_1 = b_2 = \frac{h^2}{2}$. Thus,

$$G(s) = \frac{1}{s^2} \to H(\mathrm{q}^{-1}) = \frac{h^2}{2}\frac{\mathrm{q}^{-1} + \mathrm{q}^{-2}}{(1 - \mathrm{q}^{-1})^2}$$

□

**Example: 5.4**

Sampling of a triple integrator can be made by first sampling of the expanded form

$$\frac{1}{s^2(s-\epsilon)} = \frac{-1/\epsilon}{s^2} + \frac{-1/\epsilon^2}{s} + \frac{1/\epsilon^2}{s - \epsilon}$$

and then afterwards let $\epsilon \to 0$. Thus, first sample the individual terms

$$\begin{aligned} G_1(s) &= \tfrac{1}{s^2} & \to H_1(\mathrm{q}^{-1}) &= \tfrac{h^2}{2}\tfrac{\mathrm{q}^{-1} + \mathrm{q}^{-2}}{(1 - \mathrm{q}^{-1})^2} \\ G_2(s) &= \tfrac{1}{s} & \to H_2(\mathrm{q}^{-1}) &= \tfrac{h\mathrm{q}^{-1}}{1 - \mathrm{q}^{-1}} \\ G_3(s) &= \tfrac{1}{s - \epsilon} & \to H_3(\mathrm{q}^{-1}) &= \tfrac{c\mathrm{q}^{-1}}{1 - \lambda\mathrm{q}^{-1}} \end{aligned}$$

where $\lambda = e^{\epsilon h}$ and $c = (\lambda - 1)/\epsilon$. Combine with the same linear combination

$$H(q^{-1}) = \frac{-1}{\epsilon}H_1 + \frac{-1}{\epsilon^2}H_2 + \frac{1}{\epsilon^2}H_3 = \frac{b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3}}{(1-q^{-1})^2(1-\lambda q^{-1})}$$

then use that $\lambda = e^{\epsilon h} \approx 1 + (\epsilon h) + (\epsilon h)^2/2 + (\epsilon h)^3/6 \ldots$ when finding the limits when $\epsilon \to 0$

$$\begin{array}{l} b_1 = -\frac{h^2}{2\epsilon} - \frac{h}{\epsilon^2} + \frac{\lambda-1}{\epsilon^3} \to \frac{h^3}{6} \\ b_2 = \frac{h^2}{2\epsilon}(\lambda-1) + \frac{h}{\epsilon^2}(1+\lambda) - \frac{2}{\epsilon^2}(\lambda-1) \to \frac{4h^3}{6} \\ b_3 = \frac{h^2}{2\epsilon}\lambda - \frac{h}{\epsilon^2}\lambda + \frac{\lambda-1}{\epsilon^3} \to \frac{h^3}{6} \end{array}$$

Thus,

$$G(s) = \frac{1}{s^3} \to H(q^{-1}) = \frac{h^3}{6}\frac{q^{-1} + 4q^{-2} + q^{-3}}{(1-q^{-1})^3}$$

□

## Sampling using L and Z transforms

The transfer function $H(z^{-1})$ (or operator $H(q^{-1})$) is easily found using Laplace and **Z** transform tables. The Laplace transform is defined as

$$\mathbf{L}(y) = \int_0^\infty e^{-st} y(t)dt$$

and its inverse is

$$\mathbf{L}^{-1}(Y) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} Y(s)ds$$

The **Z** transform is defined as

$$\mathbf{Z}(y) = \sum_{k=0}^{\infty} z^{-k} y(kh)$$

Consider a step input to the system. The **L** and **Z** transforms of a unit step are

$$\begin{array}{l} \mathbf{L}(1) = \int_0^\infty e^{-st}dt = \frac{1}{s} \\ \mathbf{Z}(1) = \sum_{k=0}^\infty z^{-k} = \frac{1}{1-z^{-1}} \end{array}$$

respectively. Let $G(s)$ denote the continuous-time system transfer function. Then the Laplace transform of the unit step response is $G(s)/s$. The **Z** transform of the

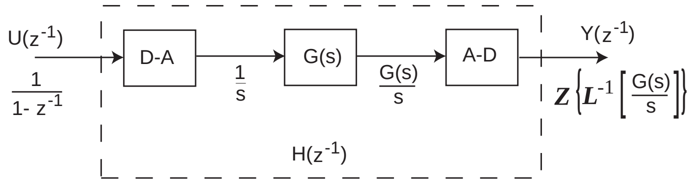



Figure 5.2: Evaluation of $H(z^{-1})$.

step response is therefore $\mathbf{Z}(\mathbf{L}^{-1}(G(s)/s))$. Dividing with the $\mathbf{Z}$ transform of the unit step input yields the transfer function. Thus,

$$H(z^{-1}) = (1 - z^{-1})\mathbf{Z}(\mathbf{L}^{-1}(\frac{G(s)}{s}))$$

as illustrated in Fig. 5.2. The formula above is not only useful when transform tables are available, but can be used to derive a residue formula. Using the definitions of $\mathbf{L}^{-1}$ and $\mathbf{Z}$ we get

$$\begin{aligned} H(z^{-1}) &= (1 - z^{-1}) \textstyle\sum_{k=0}^{\infty} z^k \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{e}^{skh} \frac{G(s)}{s} ds \\ &= (1 - z^{-1}) \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} (\textstyle\sum_{k=0}^{\infty} (z^{-1}\mathrm{e}^{sh})^k) \frac{G(s)}{s} ds \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{1 - z^{-1}}{1 - z^{-1}\mathrm{e}^{sh}} \frac{G(s)}{s} ds \\ &= \sum_{\text{Poles of} \frac{G(s)}{s}} \mathrm{Res}\{\frac{1 - z^{-1}}{1 - z^{-1}\mathrm{e}^{sh}} \frac{G(s)}{s}\} \end{aligned}$$

The last equation follows from residue calculus after extending the integration path by a large semicircle to the left. The residue of a function $f(s)$ in $s = a$, where $a$ is a pole of $f(s)$ is defined as the coefficient of $1/(s - a)$ when $f(s)$ is written in expanded form. Suppose $f(s)$ has a multiple pole of order $\ell$ in $s = a$. Then define $g(s) = (s - a)^\ell f(s)$. Since $g$ is analytic in $s = a$ it can be written as $g(s) = g(a) + g'(a)(s - a) + \ldots + \frac{g^{\ell-1}(a)}{(\ell-1)!}(s - a)^{\ell-1} + \ldots$. From this it is seen that the residue of $f(s)$ in $s = a$ is $\frac{g^{\ell-1}(a)}{(\ell-1)!}$.

**Example: 5.5**

Consider the double-integrator system $G(s) = \frac{1}{s^2}$. Sampling using the residue formula gives

$$H(z^{-1}) = Res_{s=0} \left\{ \frac{1 - z^{-1}}{1 - z^{-1}\mathrm{e}^{sh}} \frac{1}{s^3} \right\} = \frac{z - 1}{2} \frac{d^2}{ds^2} (\frac{1}{z - \mathrm{e}^{sh}}) \Big|_{s=0}$$

Successive derivations give

$$\begin{aligned}&\tfrac{d}{ds}\tfrac{1}{z-\mathrm{e}^{sh}} = \tfrac{h\mathrm{e}^{sh}}{z-\mathrm{e}^{sh}}\\&\tfrac{d^2}{ds^2}\tfrac{1}{z-\mathrm{e}^{sh}} = \tfrac{d}{ds}\tfrac{h\mathrm{e}^{sh}}{z-\mathrm{e}^{sh}} = h^2\mathrm{e}^{sh}\tfrac{z+\mathrm{e}^{sh}}{(z-\mathrm{e}^{sh})^3} \rightarrow h^2\tfrac{z+1}{(z-1)^3}, \quad s \rightarrow 0\end{aligned}$$

After substituting into the residue formula, it follows that

$$H(\mathrm{q}^{-1}) = \frac{h^2}{2}\frac{\mathrm{q}^{-1}+\mathrm{q}^{-2}}{(1-\mathrm{q}^{-1})^2}$$

□

## Sampling of state-space description

For the multiple-pole case, the sampled representation was derived from a state-space description. The states had been chosen in a particular way, representing intermediate signals between systems coupled in series. This is not necessary. In fact, any choice of state variables can be used. Consider the continuous-time system with $x \in \mathbb{R}^n$

$$\begin{aligned}&\tfrac{d}{dt}x(t) = Ax(t) + Bu(t)\\&y(t) = Cx(t)\end{aligned}$$

with transfer function $G(s) = C(sI - A)^{-1}B$. Integrating the states from $kh$ to $kh + h$ and using the fact that $u$ is piecewise constant results in

$$\begin{aligned}&x(kh+h) = \Phi x(kh) + \Gamma u(kh)\\&y(kh) = Cx(kh)\end{aligned} \qquad \left\{\begin{aligned}&\Phi = \mathrm{e}^{Ah}\\&\Gamma = \textstyle\int_0^h \mathrm{e}^{At}dtB\end{aligned}\right.$$

with transfer function $H(\mathrm{q}^{-1}) = C(\mathrm{q}I-\Phi)^{-1}\Gamma$. The matrix exponential is defined similar to the scalar case as

$$\mathrm{e}^{At} = \sum_{k=0}^{\infty}\frac{1}{k!}(At)^k$$

It is usually inconvenient to use this definition for calculation of the matrix exponential. There are several other methods. One can use the fact that the Laplace transform of $\mathrm{e}^{At}$ is $(sI - A)^{-1}$ and just look in the table for the inverse Laplace transform. Another way is to use the Cayley-Hamilton theorem, saying that $A^k$, $k > n$ can be expressed as linear combinations of $A^k$, $k = 0, \ldots, n-1$. Therefore,

$$\mathrm{e}^{At} = \sum_{k=0}^{n-1}\alpha_k(At)^k$$

The coefficients $\alpha_k$ can be found from the equation system

$$\mathrm{e}^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k (\lambda_i t)^k, \quad i = 1, \ldots, n$$

where $\lambda_i$, $i = 1, \ldots, n$ are the eigenvalues of $A$ (for multiple eigenvalues add instead derivatives of the corresponding equation to get $n$ linear independent equations). Simultaneous calculation of $\Phi$ and $\Gamma$ can be done noting that

$$\begin{array}{l} \frac{d}{dt}\Phi(t) = A\Phi(t) = \Phi(t)A \\ \frac{d}{dt}\Gamma(t) = \Phi(t)B \end{array}$$

Therefore $\Phi(t)$ and $\Gamma(t)$ satisfy

$$\frac{d}{dt}\begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

which has the solution

$$\begin{pmatrix} \Phi(h) & \Gamma(h) \\ 0 & I \end{pmatrix} = \exp\left\{\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} h\right\}$$

## 5.3 The choice of sampling period

A proper selection of the sampling period is important. If the sampling period is chosen too long it is impossible to reconstruct the continuous-time signal. If it is chosen too small the load on the computer will increase, and, in addition, numerical problems arise. It is common practice to select the sampling period related to the closed-loop system bandwidth

$$\omega_B h \approx 0.5 — 1$$

where $\omega_B$ is the frequency where the closed-loop system gain has dropped to $0.7$. Using normalized frequency as before, $\omega := \omega h$, this results in

$$|\frac{BT}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})| = 0.7, \quad \omega \in [0.5, 1]$$

If the sampling frequency, $\omega_s = 2\pi/h$, is used in the formula above it would give approximately

$$\omega_s \approx 12\omega_B$$

This is, however, only one rule among many. In some textbooks, it is recommended a factor $20$ rather than $12$ above.

## Problems with too fast sampling

Too fast sampling obviously increase the load on the computer unnecessarily. This drawback with fast sampling becomes less significant with faster computers. However, several other drawbacks remain. Since each pole $s$ of a continuous-time system maps like $\lambda = \mathrm{e}^{sh}$ by sampling, a too short $h$ results in a pole cluster close to $1$. Design on such a model is very sensitive to the precision of the parameters and to round-off errors. Consider for simplicity the characteristic equation below, perturbed with a constant term $\epsilon$,

$$(\lambda - 0.99)^4 + \epsilon = 0$$

which has the roots

$$\lambda = 0.99 + (-\epsilon)^{1/4}$$

The roots are moved from $0.99$ to a circle with origin $0.99$ and the radius $r = |\epsilon|^{1/4}$. If $\epsilon = 10^{-8}$ then $r = 10^{-2}$, i.e. the system can be unstable even if the perturbation is very small. Another problem is that the discrete-time model has numerator and denominator coefficients of different order of magnitudes. Sampling of $G(s) = 1/s$ and $G(s) = 1/s^2$ give $H(\mathrm{q}^{-1}) = h\mathrm{q}^{-1}/(1 - \mathrm{q}^{-1})$ and $H(\mathrm{q}^{-1}) = (h^2/2)(\mathrm{q}^{-1} + \mathrm{q}^{-2})/(1 - \mathrm{q}^{-1})^2$, respectively. Thus, the numerator coefficients are proportional to $h^n$, for an $n$th order system. The comparatively small numerator coefficients cause a problem if the parameters are to be identified from real data and proper scaling becomes a major issue. Also, the controller gets parameters of different order of magnitudes and special care is needed for the implementation in order to reduce round-off errors due to finite word-length representation.

Digital implementation of analog design, for example by replacing $s$ by $(\mathrm{q} - 1)/h$ or $(1 - \mathrm{q}^{-1})/h$, results only in an approximate description of the discrete-time behavior. Therefore, an even faster sampling rate than the rule of thumb above is required. However, the approximation is only improved up to a certain sampling rate and will not approach the analog design in the limit when $h$ goes to zero. The reason is because of the finite word-length of any digital implementation. Consider the updating of the integral action of a PI regulator:

$$i(kh + h) = i(kh) + e(kh) * h/T_i$$

If the sampling period is $h = 0.03$ s and the integration time is $15$ min = $900$ s, the ratio $h/T_i$ becomes $3 \cdot 10^{-5}$, which corresponds to about $15$ bits. For a small $e$ the term $eh/T_i$ might fall outside the resolution and is rounded to zero.

Integration stops. The derivative action of a PID regulator also fails when the difference between two consecutive samples is too small, resulting in round-off. A digital design has the advantage that a larger sampling period can be used, thus avoiding numerical problems, without sacrificing performance.

## Discrete design versus discretized analog design

The example below illustrates the rule of thumb for the choice of sampling period. It is shown that a much larger sampling period is appropriate for a discrete design, with anti-aliasing filter, than for a discretized analog design.

**Example: 5.6**

Consider the servo model

$$G(s) = \frac{4}{s(s+2)}$$

First, the sampling period thumb rule is illustrated for discrete designs. In Exercise 1c, in Chapter 4, a noise insensitive design is found. This is based on the sampling period $h_1 = 0.025\text{s}$ that gives the model

$$G_1(\mathrm{q}^{-1}) = \frac{B_1}{A_1} = \frac{1.23 \cdot 10^{-3}\mathrm{q}^{-1}(1 + 0.98\mathrm{q}^{-1})}{(1 - \mathrm{q}^{-1})(1 - 0.95\mathrm{q}^{-1})}$$

**Design 1:**

The pole placement is

$$A_{c1} = \prod_{k=1}^{3}(1 - \lambda_{1k}\mathrm{q}^{-1}), \quad \lambda_{1k} = 0.9, 0.93, 0.95$$

Solving the polynomial equation $A_1R_1 + B_1S_1 = A_{c1}$ and choosing $T_1 = \frac{A_{c1}}{B_1}(1)$ to get the steady-state gain one results in

$$\rightarrow \begin{cases} R_1 = 1 - 0.832\mathrm{q}^{-1} \\ S_1 = 2.931 - 2.788\mathrm{q}^{-1} \\ T_1 = 0.1435 \end{cases}$$

**Design 2:**

Try a larger sampling period $h_2 = 20h_1 = 0.5\text{s}$. This gives the sampled model

$$G_2(\mathrm{q}^{-1}) = \frac{B_2}{A_2} = \frac{0.368\mathrm{q}^{-1}(1+0.72\mathrm{q}^{-1})}{(1-\mathrm{q}^{-1})(1-0.37\mathrm{q}^{-1})}$$

The continuous-time poles corresponding to Design 1 are

$$p_k = \frac{1}{h_1}\ln\lambda_{1k}$$

Therefore, a similar design using the sampling period $h_2$ would be to choose the discrete-time poles at

$$\lambda_{2k} = e^{p_k h_2}$$

Pole placement with the desired characteristic polynomial $A_{c2} = \prod_{k=1}^{3}(1-\lambda_{2k}\mathrm{q}^{-1})$ results in

$$\rightarrow \begin{cases} R_2 = 1 + 0.257\mathrm{q}^{-1} \\ S_2 = 1.0787 - 0.396\mathrm{q}^{-1} \\ T_2 = 0.68266 \end{cases}$$

The two choices of sampling periods, $h_1$ and $h_2 = 20h_1$ are now related to the rule of thumb: $\omega'_B = \omega_B h \approx 0.5 \leftrightarrow 1$, see Fig. 5.3. The rule of thumb gives that design 2 with $h = 0.5\text{s}$ is more appropriate. It should be noted, though, that the rule of thumb assumes that an appropriate anti-aliasing filter is included. The anti-aliasing filter avoids folding down measurement noise below $\omega_B$. The Simulink system in Fig. 5.4 shows how the described example can be simulated. Measurement noise is added to the real output $y$ such that the measured output is $y_m$. In Fig. 5.5, the real and the measured responses are shown when the anti-aliasing filter is excluded and included, respectively. It is seen that, without the anti-aliasing filter, high frequency noise is folded down and appears as low frequency disturbance in the real output. In Fig. 5.6, the responses of the two design are compared, Design 1 with unnecessary fast sampling and Design 2 with sufficient sampling rate.

**Design 3:**

The discrete design above is compared to a discretized analog design. The corresponding analog pole placement is $p_k = \frac{1}{h_1}\ln\lambda_{1k}$. Pole placement design based

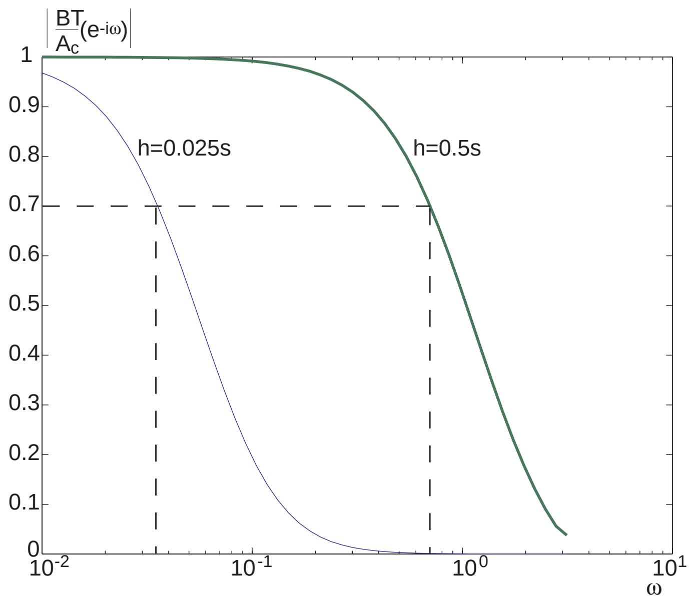



Figure 5.3: Normalized frequency response in Example 5.6; design 1 with $h = h_1 = 0.025\mathrm{s}$ (thin), design 2 with $h = h_2 = 20h_1 = 0.5\mathrm{s}$ (solid). The rule of thumb gives that design 2 with $h = 0.5\mathrm{s}$ is more appropriate.

on the continuous-time model then gives the continuous-time controller

$$\rightarrow \begin{cases} R = s + 7.169 \\ S = 3.1246s + 6.275 \end{cases}$$

Discretization using forward-difference approximation and the sampling period $h = h_3 = 0.25\mathrm{s}$ gives

$$\left.\begin{matrix} s \rightarrow \frac{\mathrm{q}-1}{h_3} \\ h_3 = 0.25 \end{matrix}\right\} \rightarrow \begin{cases} R_3 = 1 + 0.792\mathrm{q}^{-1} \\ S_3 = 3.1246 - 1.5558\mathrm{q}^{-1} \\ T_3 = 1.5688 \end{cases}$$

Using the correctly sampled model, the closed-loop poles can be calculated to

$$\lambda_{3k} = 0.63 \pm 0.19\mathrm{i}, -0.78$$

Notice the badly damped pole $-0.78$ which results in 'rippling' in the control signal, see Fig. 5.7. This is not seen in the continuous design and is therefore due to the bad approximation caused by a too large sampling period, here $h = h_3 = 0.5h_2 = 0.25\mathrm{s}$. Thus, discretized analog designs should be sampled a lot (five to ten times) faster than discrete designs. □

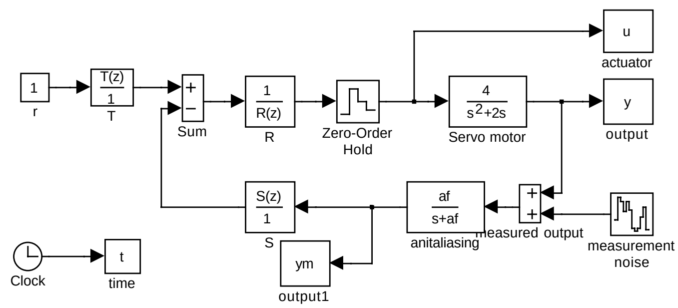



Figure 5.4: A Simulink implementation of Example 5.6.

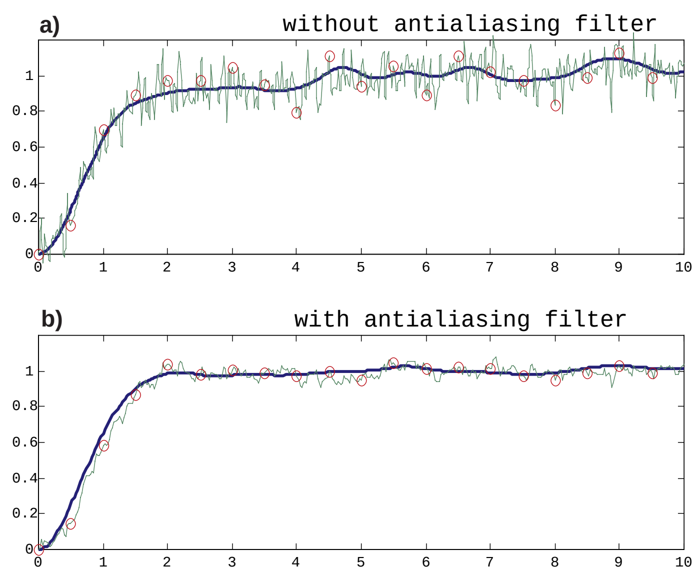



Figure 5.5: Step response for Design 2 in Example 5.6; $y$ real output (bold), $y_m$ measured output ($o$). **a)** without the anti-aliasing filter, high frequency noise is folded down and appears as low frequency disturbance, contrary to **b)** where the anti-aliasing filter is included.

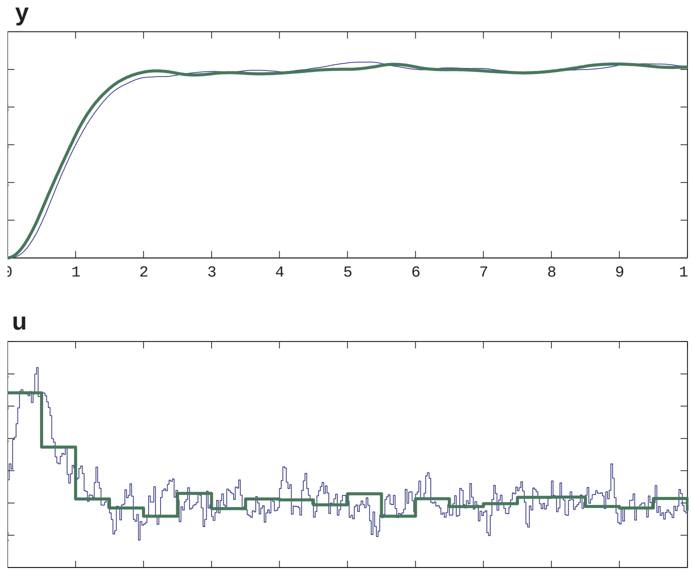



Figure 5.6: Design 1 with $h = h_1$ versus Design 2 with $h = 20h_1$ in Example 5.6.

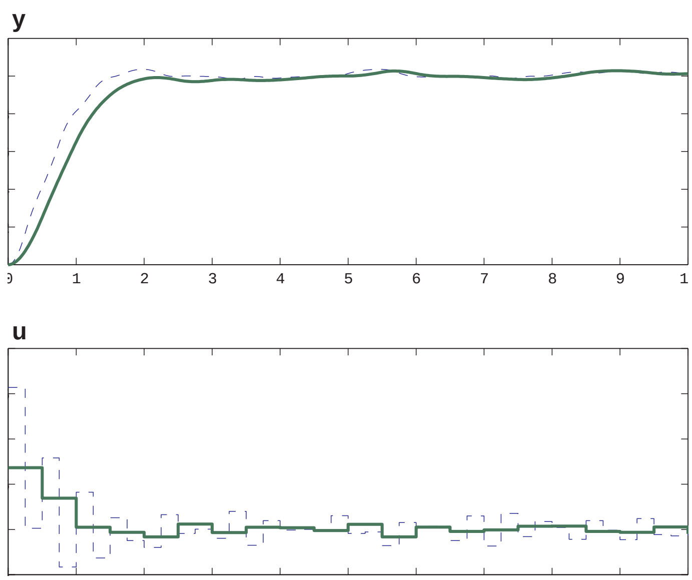



Figure 5.7: The discretized analog design (Design 3) in Example 5.6 (dashed) compared to the discrete design (Design 2).

# Exercises

**1.** Sample with period $h = 0.2$ the signal

$$y(t) = \sin 3\pi t$$

and determine

**(a)** the frequency content of the sampled signal $y(kh)$ in the interval $0 < f < f_N = 1/(2h)$;

**(b)** the periods of the continuous and sampled signals, respectively.

**(c)** If a sampled signal is periodic with period $T$, does that mean that it contains a frequency of $1/T$?

**2.** Repeat the exercise **1.** above when

$$y(t) = \sin 6t$$

**3.** An amplitude modulated signal, $u(t) = \sin(4\omega_0 t)\cos((2\omega_0 t)$, is sampled with the sampling period $h = \frac{\pi}{3\omega_0}$. What frequencies $f$, $0 \le f \le 1/(2h)$ are represented in the sampled signal?

**4.** Derive the discrete-time system corresponding to the following continuous-time systems when a zero-order-hold circuit is used

**(a)** $G(s) = \frac{s+3}{s^2+3s+2}$

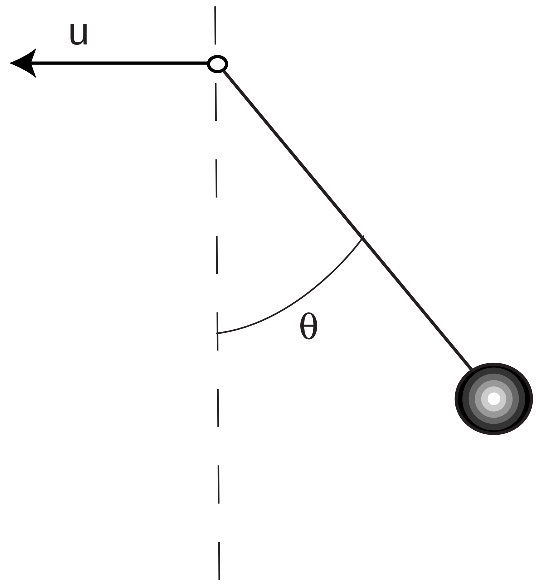
**(b)** $G(s) = \frac{1}{s^3}$

**5.** Consider the pendulum below. The acceleration of the pivot point $u$ is the input and the angle $\theta$ is the output. The acceleration of the mass $m$ in tangential direction of rotation is $a_\theta = \ell\ddot{\theta} - u\cos\theta$ where $\ell$ is the length of the pendulum. In this direction the gravitational force is $F = -mg\sin\theta$. The Newton force equation $ma_\theta = F$ therefore gives the dynamics

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = \frac{\cos\theta}{\ell}u$$

Introduce $\omega_0 = \sqrt{g/\ell}$ and $b = 1/\ell$. Linearization around $\theta = u = 0$ gives

$$\ddot{\theta} + \omega_0^2\theta = bu, \qquad G(s) = \frac{b}{s + \omega_0^2}$$




**(a)** Sample the system by using formula based on $G(s)$.

**(b)** Sample the system using formula based on the state-space description. Choose the states $x_1 = \theta\omega_0$ and $x_2 = \dot{\theta}$ to get the state-space description

$$\frac{d}{dt}x = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix} u$$
$$y = \begin{pmatrix} 1/\omega_0 & 0 \end{pmatrix} x$$

**6.** Suppose that a process with the transfer function

$$G(s) = \frac{1}{s}$$

is controlled by a discrete-time P-controller, with zero-order-hold output. However, due to computation time $\tau$, the implementation of the controller can be described by

$$u(kh) = Ke(kh - \tau), \qquad e(kh) = r(kh) - y(kh)$$

**(a)** How large are the values of the gain, $K$, for which the closed-loop system is stable if $\tau = 0$ and $\tau = h$?

**(b)** Compare this system with the corresponding continuous-time system, i.e. when there is a continuous-time proportional controller and a time delay in the process.

## Solutions

**1a.** Sampling frequency $f_s = 1/h = 5$, Nyquist frequency $f_N = f_s/2 = 5/2$. $y(t) = \sin 2\pi f t$ with frequency $f = 3/2 < f_N = 5/2 \Rightarrow$ no frequency folding below $f_N$. Frequency content in $y(kh)$: $f = 3/2$ ($\in [0, f_N]$).

**1b.** Period $T$ of $y(t) = y(t+T)$: $T = 1/f = 2/3$. Period $T_d$ of $y(kh) = y(kh + T_d)$: $T_d = 2 = 3T = 10h$. (Find integer number of $h$ that equals an integer number of $T$).

**1c.** No, there is no frequency of $1/T_d$ in $y(kh)$. Distinguish period from frequency!

**2a.** $f = 3/\pi < 5/2 = f_N$. No folding down below $f_N$: $f = 3/\pi \in [0, f_N]$.

**2b.** Period $T$ of $y(t) = y(t+T)$: $T = 1/f = \pi/3$. Period $T_d$ of $y(kh) = y(kh + T_d)$: does not exist!

**2c.** See 1c.

**3.** Write $u(t)$ as a sum of sinusoids in order to see the frequency content. Define $X(t) = \mathrm{e}^{\mathrm{i}4\omega_0 t}\mathrm{e}^{\mathrm{i}2\omega_0 t}$ and $Y(t) = \mathrm{e}^{\mathrm{i}4\omega_0 t}\mathrm{e}^{-i2\omega_0 t}$ to derive $u(t) = \frac{1}{2}\mathrm{Im}(X+Y) = \frac{1}{2}(\sin 2\omega_0 t + \sin 6\omega_0 t)$, i.e. frequencies $f_1 = \omega_0/\pi$ and $f_2 = 3\omega_0/\pi$. Since $f_2 = 2f_N$ an alias frequency is introduced at $f_a = 0$. Frequency content of $u(kh)$: $f = 0, \omega_0/\pi \in [0, f_N]$.

**4a.** Factored and expanded form $G(s) = \frac{s+3}{(s-p_1)(s-p_2)} = \frac{d_1}{s-p_1} + \frac{d_2}{s-p_2}$. Poles $p_1 = -1$, $p_2 = -2$ and $d_1 = \lim_{s\to p_1} G(s)(s-p_1) = 2$, $d_2 = \lim_{s\to p_2} G(s)(s-p_2) = -1$. $\lambda_1 = \mathrm{e}^{-h}$, $\lambda_2 = \mathrm{e}^{-2h}$, $c_1 = 1-\mathrm{e}^{-h}$, $c_2 = \frac{1}{2}(1-\mathrm{e}^{-2h})$, $b_1 = c_1 d_1 + c_2 d_2 = (1-\mathrm{e}^{-h})2 - \frac{1}{2}(1-\mathrm{e}^{-2h})$ and $b_2 = -(c_1 d_1 \lambda_2 + c_2 d_2 \lambda_1) = (1-\mathrm{e}^{-h})2\mathrm{e}^{-2h} - \frac{\mathrm{e}^{-h}}{2}(1-\mathrm{e}^{-2h})$. Sampled system $H(\mathrm{q}^{-1}) = \frac{b_1\mathrm{q}^{-1} + b_2\mathrm{q}^{-2}}{(1-\lambda_1\mathrm{q}^{-1})(1-\lambda_2\mathrm{q}^{-1})}$.

**4b.** $H(\mathrm{q}^{-1}) = C(\mathrm{q}I - \Phi)^{-1}\Gamma = \frac{h^3}{6}\frac{\mathrm{q}^{-1}+4\mathrm{q}^{-2}+\mathrm{q}^{-3}}{(1-\mathrm{q}^{-1})^3}$

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2}{2} & h & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} h \\ \frac{h^2}{2} \\ \frac{h^3}{6} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

**5a.** $G = \frac{d_1}{s-p_1} + \frac{d_2}{s-p_2}$, $p_1 = -i\omega_0$, $p_2 = i\omega_0$, $d_1 = -\frac{b}{2i\omega_0}$, $d_2 = -d_1$, $\lambda_1 = \mathrm{e}^{-\mathrm{i}\omega_0 h}$, $\lambda_2 = \mathrm{e}^{\mathrm{i}\omega_0 h}$, $c_1 = -(\mathrm{e}^{-\mathrm{i}\omega_0 h} - 1)/(\mathrm{i}\omega_0)$, $c_2 = (\mathrm{e}^{\mathrm{i}\omega_0 h} - 1)/(\mathrm{i}\omega_0)$, $b_1 = c_1 d_1 + c_2 d_2 = \frac{b}{\omega_0^2}(1 - \cos\omega_0 h)$, and $b_2 = -(c_1 d_1 \lambda_2 + c_2 d_2 \lambda_1) = b_1$. Sampled system $H(\mathrm{q}^{-1}) = \frac{b_1 \mathrm{q}^{-1} + b_2 \mathrm{q}^{-2}}{(1-\lambda_1 \mathrm{q}^{-1})(1-\lambda_2 \mathrm{q}^{-1})} = \frac{b}{\omega_0^2}(1 - \cos\omega_0 h)\frac{\mathrm{q}^{-1} + \mathrm{q}^{-2}}{1 - 2\cos\omega_0 h \mathrm{q}^{-1} + \mathrm{q}^{-2}}$.

**5b.** Calculation of $\Phi = \mathrm{e}^{Ah}$ and $\Gamma = \int_0^h \mathrm{e}^{As} ds B$ can be done by introducing

$$X(t) = \begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix}, \qquad M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\frac{d}{dt} X(t) = X(t)M, \quad \rightarrow X(t) = X(0)\mathrm{e}^{Mt} = \mathrm{e}^{Mt}$$

The solution is of the form $X(h) = \alpha_0 I + \alpha_1 Mh + \alpha_2 (Mh)^2$ where $\alpha_k$, $k = 0, 1, 2$ are calculated from corresponding equations when $M$ is replaced by its eigenvalues ($\det(\lambda_i I - M) = 0$). These are $\lambda_1 = -\mathrm{i}\omega_0$, $\lambda_2 = \mathrm{i}\omega_0$, $\lambda_3 = 0$:

$$\left.\begin{array}{l} \mathrm{e}^{-i\omega_0 h} = \alpha_0 + \alpha_1(-\mathrm{i}\omega_0 h) + \alpha_2(-\mathrm{i}\omega_2 h)^2 \\ \mathrm{e}^{\mathrm{i}\omega_0 h} = \alpha_0 + \alpha_1(\mathrm{i}\omega_0 h) + \alpha_2(\mathrm{i}\omega_2 h)^2 \\ 1 = \alpha_0 \end{array}\right\} \quad \rightarrow \left\{\begin{array}{l} \alpha_0 = 1 \\ \alpha_1 h = \frac{1}{\omega_0}\sin\omega_0 h \\ \alpha_2 h^2 = \frac{1}{\omega_0^2}(1 - \cos\omega_0 h) \end{array}\right.$$

This gives

$$X(h) = \begin{pmatrix} \Phi(h) & \Gamma(h) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \cos\omega_0 h & \sin\omega_0 h & \vdots & \frac{b}{\omega_0}(1 - \cos\omega_0 h) \\ -\sin\omega_0 h & \cos\omega_0 h & \vdots & \frac{b}{\omega_0}\sin\omega_0 h \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \vdots & 1 \end{pmatrix}$$

Alternatively, using the Laplace transform,

$$\Phi = \mathrm{e}^{Ah} = \mathbf{L}^{-1}(sI - A)^{-1} = \mathbf{L}^{-1}\begin{pmatrix} \frac{s}{s^2+\omega_0^2} & \frac{\omega_0}{s^2+\omega_0^2} \\ -\frac{\omega_0}{s^2+\omega_0^2} & \frac{s}{s^2+\omega_0^2} \end{pmatrix} = \begin{pmatrix} \cos\omega_0 h & \sin\omega_0 h \\ -\sin\omega_0 h & \cos\omega_0 h \end{pmatrix}$$

$$\Gamma = \int_0^h \Phi(s) B ds = b\int_0^h \begin{pmatrix} \sin\omega_0 s \\ \cos\omega_0 s \end{pmatrix} ds = \frac{b}{\omega_0}\begin{pmatrix} 1 - \cos\omega_0 h \\ \sin\omega_0 h \end{pmatrix}$$

**6a.** For $\tau = 0$: $\mathsf{A}_c = 1 - (1 - Kh)\mathrm{q}^{-1} \Rightarrow 0 < K < 2/h$ and for $\tau = h$: $\mathsf{A}_c = 1 - \mathrm{q}^{-1} + Kh\mathrm{q}^{-2} \Rightarrow 0 < K < 1/h$.

**6b.** $Ac = s + K\mathrm{e}^{-s\tau}$. The Nyquist curve crosses negative real axis for $\omega_0$: $-\pi = \arg K\mathrm{e}^{-\mathrm{i}\omega_0\tau}/(\mathrm{i}\omega_0) = -\pi/2 - \omega_0\tau \Rightarrow \omega_0 = \pi/(2\tau)$. Stability is lost when $|K\mathrm{e}^{-\mathrm{i}\omega_0\tau}/(\mathrm{i}\omega_0)| = 1 \Rightarrow K = \omega_0 = \pi/(2\tau)$, giving $0 < K < \pi/(2\tau)$.

# Chapter 6

# Identification

System identification refers to the problem of estimating the parameters of the model, here the coefficients of the difference equation, given samples of the process input and output. The simplest formulation of the identification problem is to minimize the squares of the equation error which gives a linear least-squares problem with analytical solution. This solution can also be constructed recursively, thus reducing the need for data storage and opening possibilities for on-line use such as adaptive control. In some cases, for example when the process is unstable, badly damped or nonlinear, it is necessary to close the loop before trying to estimate the process model. The least-square estimate can still be useful when estimating from closed-loop data, conditioned that an appropriate data filter is used.

## 6.1 The identification problem

Plant model to be identified describes the real process input $u$ and output $y$ according to

$$A(\theta)y(k) = B(\theta)u(k) + \varepsilon_\theta(k) \tag{6.1}$$

where $\varepsilon$ is the equation error that includes what is not modeled (nonlinearities and unmodeled dynamics). The linear part is described by

$$\begin{aligned}
&A(\theta) = 1 + a_1\mathrm{q}^{-1} + \ldots a_{\deg A}\mathrm{q}^{-\deg A} \\
&B(\theta) = b_d\mathrm{q}^{-\mathrm{d}} + \ldots b_{\deg B}\mathrm{q}^{-\deg B} \\
&\theta = \begin{pmatrix} a_1 & \ldots & a_{\deg A} & b_d & \ldots & b_{\deg B} \end{pmatrix}^T
\end{aligned}$$

where all parameters have been collected in a vector $\theta$. Rewrite (6.1) as linear regression

$$y(k) = \varphi(k)^T\theta + \varepsilon_\theta(k) \tag{6.2}$$

where

$$\varphi(k) = \begin{pmatrix} -y(k-1) \\ \vdots \\ -y(k-\deg A) \\ u(k-d) \\ \vdots \\ u(k-\deg B) \end{pmatrix}$$

*The identification problem:* Find estimate $\hat{\theta}$ such that $\varepsilon_{\hat{\theta}}(k)$ is 'small' in some sense.

## 6.2 The least-squares problem

Consider $N$ observations ($N > \dim\theta$) of (6.2)

$$\begin{aligned} y(1) &= \varphi(1)^T\theta + \varepsilon_\theta(1) \\ y(2) &= \varphi(2)^T\theta + \varepsilon_\theta(2) \\ &\vdots \\ y(N) &= \varphi(N)^T\theta + \varepsilon_\theta(N) \end{aligned}$$

In matrix notation this can be written as

$$Y = \Phi\theta + \varepsilon_\theta$$

where

$$Y = \begin{pmatrix} y(1) \\ \vdots \\ y(N) \end{pmatrix}, \Phi = \begin{pmatrix} \varphi(1)^T \\ \vdots \\ \varphi(N)^T \end{pmatrix}, \varepsilon_\theta = \begin{pmatrix} \varepsilon_\theta(1) \\ \vdots \\ \varepsilon_\theta(N) \end{pmatrix}$$

The criterion is chosen as the sum of squares of the equation errors

$$V(\theta) = \frac{1}{2}\varepsilon_\theta^T\varepsilon_\theta = \frac{1}{2}\sum_{k=1}^{N}\varepsilon_\theta(k)^2$$

The least-squares estimate is then defined to be

$$\hat{\theta} = \arg\min V(\theta)$$

## The least-squares estimate

The least-squares estimate has an explicit solution. This is shown below. Reconsider the equation system

$$Y = \Phi\theta + \varepsilon_\theta$$

The criterion is then

$$V(\theta) = \frac{1}{2}\varepsilon_\theta^T \varepsilon_\theta = \frac{1}{2}(Y - \Phi\theta)^T (Y - \Phi\theta)$$

At the minimum, the gradient is zero

$$0 = \frac{dV(\theta)}{d\theta} = -Y^T\Phi + \theta^T(\Phi^T\Phi)$$

giving the least-squares estimate

$$\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y$$

**Example: 6.1**

Given $N = 1000$ samples of $y$ and $u$ collected from system

$$y(k) - 0.9y(k-1) = 0.1u(k-1) + e(k)$$

where $u$ and $e$ are independent white noise, both with zero mean $E[u(k)] = E[e(k)] = 0$, and unit variance $E[u^2(k)] = E[e^2(k)] = 1$. The true parameter vector is

$$\theta_0 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -0.9 \\ 0.1 \end{pmatrix}$$

The least-squares estimate is calculated to

$$\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y = \begin{pmatrix} -0.897 \\ 0.090 \end{pmatrix}$$

where

$$\Phi = \begin{pmatrix} -y(1) & u(1) \\ \vdots & \vdots \\ -y(N-1) & u(N-1) \end{pmatrix}, \quad Y = \begin{pmatrix} y(2) \\ \vdots \\ y(N) \end{pmatrix}$$

□

### Bias of the least-squares estimate

Suppose the true plant is

$$y(k) = \varphi(k)^T \theta_0 + e(k), \quad k = 1, \ldots, N$$

or in matrix form

$$Y = \Phi \theta_0 + e$$

Least-squares estimate

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T (\Phi \theta_0 + e) = \theta_0 + (\Phi^T \Phi)^{-1} \Phi^T e$$

If $e$ zero mean and uncorrelated with $\Phi$

$$E[\hat{\theta}] = \theta_0$$

**Example: 6.2**

Reconsider Example 6.1, but suppose that $E[e(k)] = 1$. Then the least-squares estimate is biased

$$\hat{\theta} = \begin{pmatrix} -0.994 \\ 0.079 \end{pmatrix}$$

Notice that the bias forces $A(\mathrm{q}^{-1})$ to include the factor $1 - \mathrm{q}^{-1}$ in order to eliminate the bias, which is not modeled by any other terms of the model. □

## 6.3 The recursive least-squares algorithm

It is convenient to formulate the least-square estimate recursively such that it can be found on-line. From the definition of $\Phi$ and $Y$, it follows that

$$\hat{\theta}(k) = [\sum_{t=1}^{k} \varphi(t)\varphi(t)^T]^{-1} \sum_{t=1}^{k} \varphi(t) y(t)$$

Introduce the notation

$$P(k) = [\sum_{t=1}^{k} \varphi(t)\varphi(t)^T]^{-1}$$

Then since

$$P^{-1}(k) = P^{-1}(k-1) + \varphi(k)\varphi(k)^T \tag{6.3}$$

it follows that

$$\begin{aligned}\hat{\theta}(k) &= P(k)[\textstyle\sum_{t=1}^{k-1}\varphi(t)y(t)+\varphi(k)y(k)]\\ &= P(k)[P^{-1}(k-1)\hat{\theta}(k-1)+\varphi(k)y(k)]\\ &= \hat{\theta}(k-1)+P(k)\varphi(k)[y(k)-\varphi(k)^T\hat{\theta}(k-1)]\end{aligned}$$

or simply

$$\begin{cases}\hat{\theta}(k)=\hat{\theta}(k-1)+K(k)\varepsilon(k)\\ K(k)=P(k)\varphi(k)\\ \varepsilon(k)=y(k)-\varphi(k)^T\hat{\theta}(k-1)\end{cases}$$

Notice that $\hat{y}(k)=\varphi(k)^T\theta(k-1)$ is the prediction of the output given data up to $k-1$. Therefore, $\varepsilon$ can here be called the prediction error.

## Simplifications

In (6.3) a costly matrix inversion is needed before implementing the update gain $K$ above. This can be avoided by using the matrix inversion lemma below.

**Lemma 1 (Matrix inversion lemma)** *Provided the inverses below exist it holds that*

$$(A+BCD)^{-1}=A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

*Proof.*

$$\begin{aligned}&(A+BCD)(A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1})\\ &=I+BCDA^{-1}-(B+BCDA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\\ &=I+BCDA^{-1}-BC(C^{-1}+DA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}=I\end{aligned}$$

■

In (6.3) identify $P(k-1)$, $\varphi(k)$, $1$, and $\varphi(k)^T$ with $A$, $B$, $C$ and $D$ in the lemma. Then (6.3) turns into

$$P(k)=P(k-1)-\frac{P(k-1)\varphi(k)\varphi(k)^TP(k-1)}{1+\varphi(k)^TP(k-1)\varphi(k)}$$

Since the multiplication $P(k-1)\varphi(k)$ appears three times, an auxiliary variable can be introduced to avoid repeating the same multiplication. Further simplifications can then be made since now

$$K(k)=P(k)\varphi(k)=P(k-1)\varphi(k)/[1+\varphi(k)^TP(k-1)\varphi(k)]$$

which can also be expressed in the same auxiliary variable. Finally, the *recursive least-squares algorithm:*

$$\text{auxiliary variables:} \quad \begin{cases} n(k) = P(k-1)\varphi(k) \\ d(k) = 1 + \varphi(k)^T n(k) \\ K(k) = n(k)/d(k) \\ \varepsilon(k) = y(k) - \varphi(k)^T \hat{\theta}(k-1) \end{cases}$$

$$\text{state updates:} \quad \begin{cases} \hat{\theta}(k) = \hat{\theta}(k-1) + K(k)\varepsilon(k) \\ P(k) = P(k-1) - n(k)n(k)^T/d(k) \end{cases}$$

Initial conditions need also to be specified, e.g. $\hat{\theta}(0) = 0$, $P(0) = 10^4 \cdot I$. The matrix $P(0)$ has an interpretation of covariance matrix of the initial estimate $\hat{\theta}(0)$. If the latter is to be considered completely unknown, $P(0)$ should be chosen large.

**Example: 6.3**

Recursive least-square estimation for Example 6.1 is shown in Fig. 6.1 where the initial states are

$$\hat{\theta}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P(0) = 10^4 \cdot I_{2\times 2}$$

□

## 6.4 Closed-loop identification

When using the least-squares method in closed loop it is important to first filter the data through a filter what removes irrelevant information above the designed bandwidth. This improved the estimates considerably. Below, the appropriate data filter for closed-loop identification is derived. But first compare to the open loop case. The open-loop equation error $\varepsilon$ is

$$\varepsilon = Ay - Bu$$

and the open-loop identification problem is: $\min V(\theta) = \sum \varepsilon^2(k)$. Now, the closed-loop response is

$$y = \underbrace{\frac{BT}{A_c} r}_{y_m} + \frac{R}{A_c}\varepsilon$$

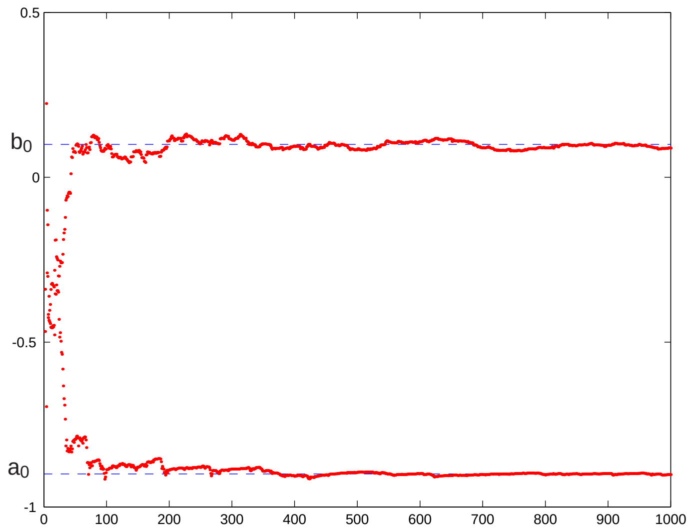



Figure 6.1: The recursive least-squares estimation in Example 6.3

The closed-loop unmodeled response $e_u$ is defined as

$$e_u = y - y_m = \frac{R}{A_c}\varepsilon = \frac{R}{A_c}(Ay - Bu) = A\underbrace{(\frac{R}{A_c}y)}_{y_F} - B\underbrace{(\frac{R}{A_c}u)}_{u_F}$$

But this looks the same as for the open-loop case with the only difference that $u$ and $y$ are first filtered. Thus, the appropriate data filter is $\frac{R}{A_c}$ and the closed-loop identification problem is: $\min V(\theta) = \sum e_u^2(k)$.

**Example: 6.4**

Consider the servo model sampled with period $h = 0.5$

$$G(s) = \frac{4}{s(s+2)} \rightarrow H(\mathrm{q}^{-1}) = \frac{0.3679\mathrm{q}^{-1} + 0.2642\mathrm{q}^{-2}}{1 - 1.3679\mathrm{q}^{-1} + 0.3679\mathrm{q}^{-2}}$$

controlled in closed loop with the controller

$$\begin{cases} R = 1 + 0.2567\mathrm{q}^{-1} \\ S = 1.0787 - 0.3961\mathrm{q}^{-1} \\ T = 0.68266 \end{cases}$$

White noise is added to the output. The effect of the data filter $F = \frac{R}{A_c}$ is illustrated in Table 6.1

□

| | $\theta_0$ | $\hat{\theta}$ | $\hat{\theta}_F$ |
|---|---|---|---|
| $a_1$ | $-1.3679$ | $-1.2099$ | $-1.3552$ |
| $a_2$ | $0.3679$ | $0.2178$ | $0.3543$ |
| $b_1$ | $0.3679$ | $0.3735$ | $0.3744$ |
| $b_2$ | $0.2642$ | $0.2881$ | $0.2707$ |

Table 6.1: Effect of the data filter $F = \frac{R}{A_c}$ when estimating during closed-loop operation. The true parameter vector is $\theta_0$. The filtered least-squares estimate is $\hat{\theta}_F$ and the unfiltered is $\hat{\theta}$.

# Chapter 7

# Practical design criteria

The exists controller design methods for almost any imaginable design criterion. Typical criteria are various norms in time- and in frequency domain. Not all of them make sense in practice. For example, minimum variance control, which has a time domain (2-norm) criterion, can lead to unbounded (maximum variance) control signals. Robust design methods, that take into account model uncertainties in the design, are typically using frequency domain criteria. Classical loop shaping is an example of this. A practical design must involve analysis of many aspects of the controlled system. For example, the amplitude and phase margins should be analyzed, or, even better, taken into account in the design. It is shown here how pole placement technique can be used to improve robustness margins.

## 7.1 Classical robustness margins

The *amplitude margin* $A_m$ and the *phase margin* $\phi_m$ are the classical robustness margins that were used already by the pioneers Bode and Nyquist in the forties. They have been used since then and are known to make sense. Their definitions are illustrated for a Nyquist curve in Fig. 7.1. The amplitude (or gain) margin is the gain the loop transfer can be amplified with before stability is lost. The phase margin is the amount of phase loss of the loop that can be tolerated before stability is lost. Phase loss is related to time delay. The phase margin can therefore be translated to a *delay margin* $\tau_d = \phi_m/\omega_c$, where $\omega_c$ is the cross-over frequency (which is where the loop gain is $|L(\mathrm{e}^{-\mathrm{i}\omega_c}| = 1$).

While the amplitude and phase margins are accounted for because of uncertainties of the plant, the delay margin is also of interest because of the uncertainty

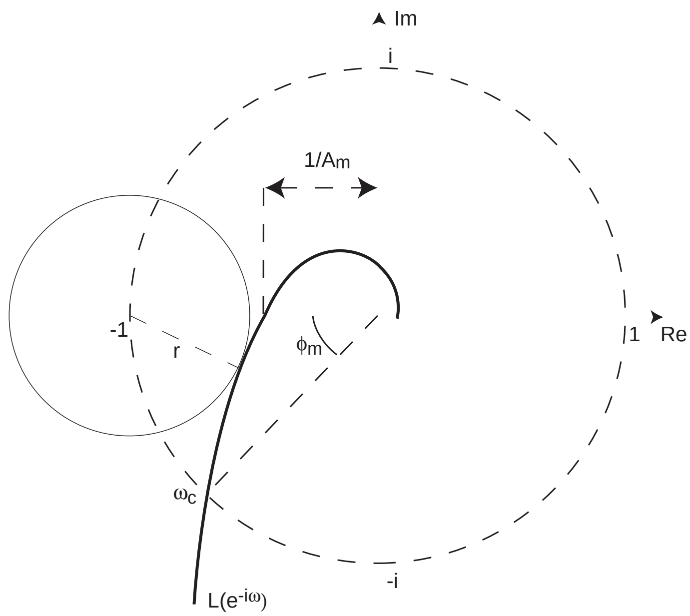



Figure 7.1: Nyquist curve of loop transfer $L = \frac{BS}{AR}$ and definitions of amplitude margin $A_m$ and phase margin $\phi_m$. Delay margin is $\tau_d = \frac{\phi_m}{\omega_c}$.

in the controller implementation. In a computer-controlled system the control signal is supposed to be generated at the same time instant as the sampling. This is, however, impossible due to the calculation time required for the control law. Often the calculation time is small compared to the sampling interval, but can of course be disturbed by other tasks occupying the computer, plotting of data for example. The delay margin is also relevant for resonant systems having many cross-over frequencies. It is not always the crossing with the smallest phase margin that determines the delay margin. This is shown in an example later (Fig. 7.2).

## 7.2 Sensitivity functions

### The output sensitivity function

Consider a simple feedback system with the plant

$$y = Pu$$

and the controller

$$u = C(r - y)$$

which in closed loop is

$$y = Hr, \qquad H = \frac{CP}{1 + CP}$$

In classical control *the sensitivity function*, or as we will call it here *the output sensitivity function*, $\mathcal{S}_y$, is defined as the relative precision in the closed loop to variations in the plant

$$\mathcal{S}_y = \frac{dH/H}{dP/P} = \frac{1}{1 + CP}$$

The index $y$ is used here partly to distinguish from the $S$ polynomial of the controller and partly to give another meaning to the sensitivity function, described below. In polynomial form, the plant and the controller are

$$\begin{aligned} Ay &= Bu + C\varphi \\ Ru &= -Sy + Tr \end{aligned}$$

Choosing $C = A$ to make $\varphi$ an additive output disturbance results in

$$y = \mathcal{S}_y \varphi, \qquad \mathcal{S}_y = \frac{AR}{A_c}$$

In order to reduce low frequency disturbances, $\mathcal{S}_y$ should be small at low frequencies. But it is also important to bound $\mathcal{S}_y$ at other frequencies. Consider the largest disc centered at $-1$ with a radius $r$ such that the disc touches the Nyquist curve, see Fig 7.1. It follows that

$$\max_\omega |\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})| = \max \frac{1}{|1 + L(\mathrm{e}^{-\mathrm{i}\omega})|} = \frac{1}{\min |1 + L(\mathrm{e}^{-\mathrm{i}\omega})|} = \frac{1}{r}$$

Thus, the Bode amplitude curve of $\mathcal{S}_y$ gives information about guaranteed bounds on the robustness margins $A_m$ and $\phi_m$. From Fig 7.1 it is clear that

$$\begin{cases} A_m \geq \frac{1}{1-r} \\ \phi_m \geq 2\arcsin(\frac{r}{2}) \end{cases}$$

By requiring $\max|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})| < 2$, the system will have at least the robustness margins $A_m \geq 2$ and $\phi_m \geq 29^o$.

It can also be interesting to consider another output sensitivity function that relates the output response to a disturbance acting additive to the input, i.e. by putting $C = B$. This is

$$\frac{BR}{A_c}$$

For example, the system can have a badly damped pole-pair and still $\max|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$ is well bounded due to a cancellation between $A$ and $A_c$ (like in internal model control design). The same cancellation will not appear between $B$ and $A_c$. Therefore, the sensitivity function above will show a nasty peak, revealing that an undamped response can be expected if a disturbance enters at the process input.

### The input sensitivity function

The main drawback with feedback is the feedback of measurement noise. Let $\varphi$ be measurement noise added to the output by putting $C = A$. Then the measurement noise influences the control action as

$$u = \mathcal{S}_u\varphi, \quad \mathcal{S}_u = -\frac{AS}{A_c}$$

In order to reduce noise feedback, $|\mathcal{S}_u(\mathrm{e}^{-\mathrm{i}\omega})|$ should be bounded at high frequencies.

## 7.3 Shaping of sensitivity functions by pole placement

When a sensitivity function, for example $|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$, is plotted against frequency $\omega \in [0, \pi]$ (where $\omega$ has been normalized by multiplication of the sampling period) $\omega$ also gets the meaning of 'phase'. A local maximum at a frequency $\omega$ can easily be identified with a complex pole-pair with phase $\omega$. This can be realized by

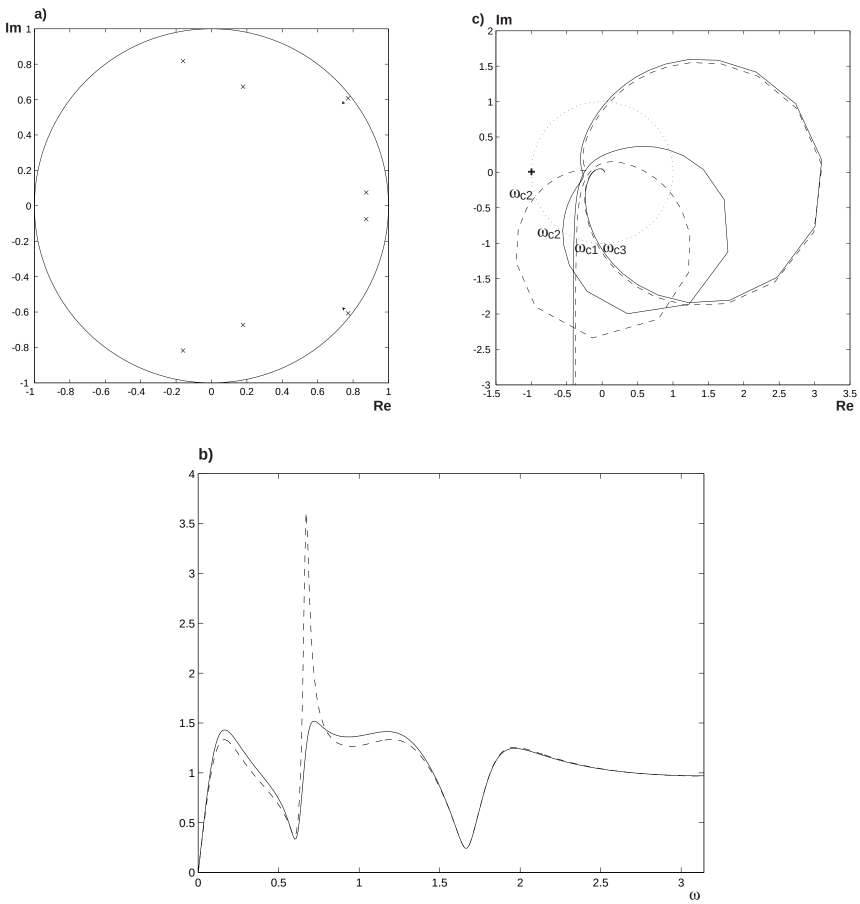



Figure 7.2: **a.** Closed-loop poles. **b.** $|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$, $\omega \in [0, \pi]$. **c.** Nyquist curves. The solid curves correspond to *after* the move of the pole-pair shown in **a**.

comparing the phases of closed-loop poles in Fig. 7.2a to the locations of local maxima of $|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$ in Fig. 7.2b.

Once the pole or pole-pair has been identified as the critical one, that causes a nasty peak, it is moved towards the origin. The peak of the sensitivity function is then reduced, see Fig 7.2ab. This can be done successively for many critical poles until the magnitude of the sensitivity function is shaped as desired.

Because of the relation between $\max|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$ and the robustness margins $A_m$ and $\phi_m$, it is interesting to see how the minimization of $\max|\mathcal{S}_y(\mathrm{e}^{-\mathrm{i}\omega})|$ effects the Nyquist curve, see Fig. 7.2c. Notice how the resonance bubble is moved away from the critical point $-1$.

This example also shows that the delay margin is of particular importance for resonant systems. Here, there are three cross-over frequencies and corresponding phase margins that need to be evaluated in order to determine $\tau_{dk} = \phi_{mk}/\omega_{ck}$, $k = 1, 2, 3$, whereafter the delay margin can be calculated as $\tau_d = \min(\tau_{d1}, \tau_{d2}, \tau_{d3})$. In this example, it is not the same crossing that determines the delay margin before and after the move of the pole-pair!

In a similar fashion, the input sensitivity function $|\mathcal{S}_u(\mathrm{e}^{-\mathrm{i}\omega})|$ should be reduced at high frequencies in order to avoid unnecessary noise feedback. This is done primarily by moving poles corresponding to high frequencies (with phases closer to $\pi$).

# Exercises

**1.** The closed-loop poles and the corresponding 'output-sensitivity function' $|\frac{AR}{A_c}|$ are shown in Fig. 7.2a and b, respectively. It is illustrated how a peak in the sensitivity function is reduced by moving the complex pole-pair that is mostly responsible for the peak.

**a.** Other pole-pairs are mostly influencing other local maxima of $|\frac{AR}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})|$. Identify these by associating pole-pair to local maxima.

**b.** Estimate the amplitude and phase margins, $A_m$ and $\varphi_m$, from $|\frac{AR}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})|$.

**c.** Estimate $A_m$ and $\varphi_m$ from the Nyquist curves in Fig. 7.2c.

**d.** The cross-over (normalized) frequencies are $\omega_{c1} = 0.07$, $\omega_{c2} = 0.66$ and $\omega_{c3} = 1.76$ for the dashed curve, and corresponding angles to the negative real axis are $\varphi_{m1} = 1.19$ (rad), $\varphi_{m2} = 0.34$ and $\varphi_{m3} = 1.51$. Corresponding values for the solid curve are $\omega_{c1} = 0.068$, $\omega_{c2} = 0.66$, $\omega_{c3} = 1.76$, $\varphi_{m1} = 1.17$, $\varphi_{m2} = 0.97$ and $\varphi_{m3} = 1.53$. What is the delay margin?

## Solutions

**1a.** The normalized frequency $\omega$ at a local maximum of $|\frac{AR}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})|$ is approximately equal to the argument of a complex pole-pair.

**1b.** **Before move:** $\max |\frac{AR}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})| = 3.61$ at $\omega = 0.67$. Take as estimate $\hat{\omega}_c = 0.67$. The Nyquist curve avoids a disc at $-1$ of radius $r = 1/3.61$. Estimation of the margins: $A_m \geq 1/(1-r) = 1.38$ and $\varphi_m = 2\arcsin(r/2) = 0.28$ ($16^o$).

**After move**: $\max|\frac{AR}{A_c}(\mathrm{e}^{-\mathrm{i}\omega})| = 1.52$ at $\omega = 0.72$. Take as estimate $\hat{\omega}_c = 0.72$. The Nyquist curve avoids a disc at $-1$ of radius $r = 1/1.52$. Estimation of the margins: $A_m \geq 1/(1-r) = 2.92$ and $\varphi_m = 2\arcsin(r/2) = 0.67$ ($38^o$).

**1c. Before move:** The Nyquist curve crosses the negative real axis at $-0.45 = -1/A_m$, i.e. $A_m = 1/0.45 = 2.2$. The Nyquist curve crosses the unit circle at an angle of $\varphi_m = 0.34$ ($19^o$) to the negative real axis.

**After move:** The Nyquist curve crosses the negative real axis at $-0.37 = -1/A_m$, i.e. $A_m = 1/0.37 = 3.6$. The Nyquist curve crosses the unit circle at an angle of $\varphi_m = 0.97$ ($56^o$) to the negative real axis.

**1d.** Calculate first $\tau_k = \varphi_{mk}/\omega_{ck}$ for $k = 1, 2, 3$. Before move of poles (dashed curve) these are: $\tau_{d1} = 17$, $\tau_{d2} = 0.52$ and $\tau_{d3} = 0.86$, which gives the delay margin $\tau_d = \min[\tau_{d1}, \tau_{d2}, \tau_{d3}] = \tau_{d2} = 0.52$ fractions of the sampling period. After the move of poles (solid curve): $\tau_{d1} = 18$, $\tau_{d2} = 1.47$ and $\tau_{d3} = 0.87$, which gives $\tau_d = \min[\tau_{d1}, \tau_{d2}, \tau_{d3}] = \tau_{d3} = 0.87$ fractions of the sampling period.

# Chapter 8

# Optimal disturbance rejection and tracking

Optimal asymptotic rejection of disturbances and tracking errors can be treated in a unified way using the *internal model principle*. This notion means that a model of the disturbance (or the tracking error) is included internally in the controller structure in such a way that the disturbance is *annihilated*. It results in an asymptotic rejection of all disturbances (or tracking errors) that can be described by the same model structure.

Describing signals as models is here fundamental. First, the concept of annihilation is introduced and applied in closed-loop design for disturbance rejection. Then, tracking of reference signals are put into the annihilation framework. This results in another polynomial equation, from which the $T$ polynomial is calculated.

## 8.1 Disturbance models

Many signals can be specified by models of the form

$$\Phi(\mathrm{q}^{-1})\varphi(k) = N(\mathrm{q}^{-1})\delta(k) \tag{8.1}$$

where the initial conditions are all zero $\varphi(-1) = \ldots = \varphi(-\deg\Phi) = 0$ and the unit pulse $\delta(k) = 1, (= 0)$, $k = 0, (\neq 0)$. The polynomials are defined as

$$\begin{aligned} \Phi(\mathrm{q}^{-1}) &= 1 + f_1\mathrm{q}^{-1} + \ldots + f_{\deg\Phi}\mathrm{q}^{-\deg\Phi} \\ N(\mathrm{q}^{-1}) &= n_0 + n_1\mathrm{q}^{-1} + \ldots + n_{\deg N}\mathrm{q}^{-\deg N}, \qquad \deg N < \deg\Phi \end{aligned}$$

| $\varphi(k)$ | $\Phi(\mathrm{q}^{-1})$ |
|---|---|
| $a$ | $1-\mathrm{q}^{-1}$ |
| $a+bk$ | $(1-\mathrm{q}^{-1})^2$ |
| $a+bk+ck^2$ | $(1-\mathrm{q}^{-1})^3$ |
| $a\sin(\omega k+b)$ | $1-\mathrm{q}^{-1}2\cos\omega+\mathrm{q}^{-2}$ |
| $\varphi(k)=\varphi(k-T)$ | $1-\mathrm{q}^{-T}$ |

Table 8.1: Signals $\varphi(k)$ with corresponding annihilation polynomial $\Phi(\mathrm{q}^{-1})$. Note that the parameters $a$, $b$ and $c$ are arbitrary.

Given $\Phi$ and $\varphi(k)$, for $k=0,\dots,\deg\Phi$, the polynomial $N$ can be calculated from the set of equations

$$
\begin{aligned}
&\varphi(0)=n_0\\
&\varphi(1)+f_1\varphi(0)=n_1\\
&\varphi(2)+f_1\varphi(1)+f_2\varphi(0)=n_2\\
&\vdots\\
&\varphi(\deg\Phi)+f_1\varphi(\deg\Phi-1)\dots f_{\deg\Phi}\varphi(0)=n_{deg\Phi}=0 \quad (\deg N<\deg\Phi)\\
&\varphi(k)+f_1\varphi(k-1)+\ldots+f_{\deg\Phi}\varphi(\deg\Phi)=\Phi(\mathrm{q}^{-1})\varphi(k)=0, \quad k\geq\deg\Phi
\end{aligned}
$$

Hence, it holds that

$$\Phi(\mathrm{q}^{-1})\varphi(k)=0, \qquad k\geq\deg\Phi(\mathrm{q}^{-1}) \tag{8.2}$$

which is independent of $N$ ($\deg N<\deg\Phi$). Thus, while $N$ defines the specific signal, the polynomial $\Phi$ defines a whole class of signals for which (8.2) holds after a transient. This is called *annihilation* or *absorption*, and $\Phi$ is called annihilation polynomial. It annihilates (or absorbs) a set of signals whereafter nothing remains after a finite time. The signal $\varphi(k)$ represents a nonzero, bounded disturbance, which means that $\Phi(\mathrm{q}^{-1})$ has zeros on the unit circle. In Table 8.1 a list of different signals $\varphi$ with corresponding annihilation polynomial is given. Notice that the parameters $a$, $b$ and $c$ are arbitrary.

**Example: 8.1**

Consider the signal $\varphi(k)$ in Fig. 8.1a, consisting of piecewise parabolas of the form $\varphi(k)=a+bk+ck^2$ with parameters $a$, $b$ and $c$ changing every 10 sample. The annihilation polynomial $\Phi=(1-\mathrm{q}^{-1})^3$ eliminates each parabolic signal after

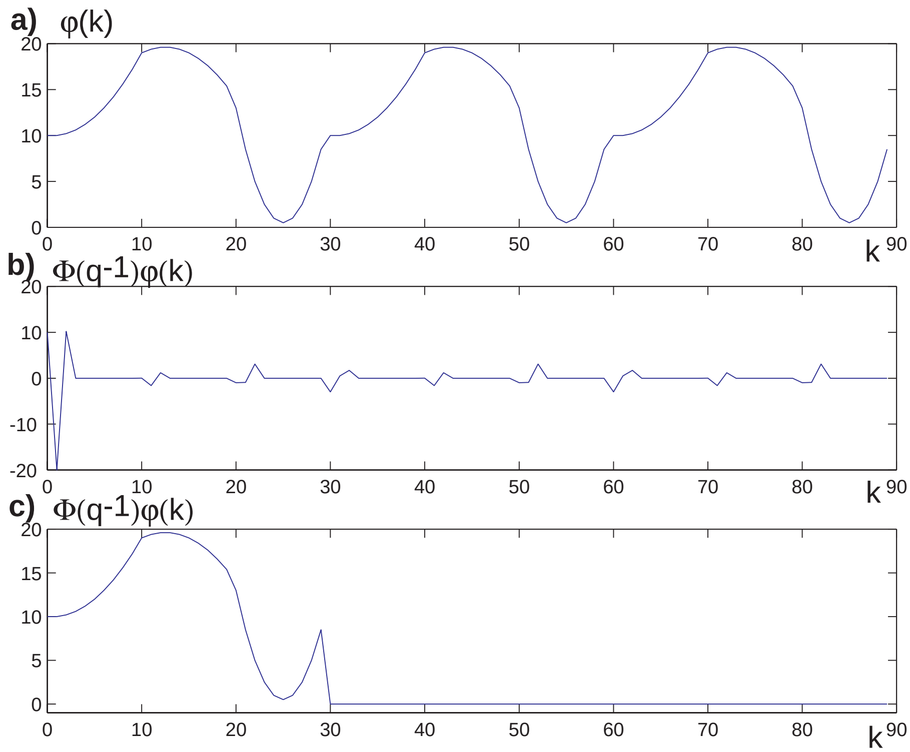



Figure 8.1: **a)** $\varphi(k)$, **b)** $\Phi(\mathrm{q}^{-1})\varphi(k)$, $\Phi(\mathrm{q}^{-1}) = (1 - \mathrm{q}^{-1})^3$, **c)** $\Phi(\mathrm{q}^{-1})\varphi(k)$, $\Phi(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-30}$

a transient of $3$ ($= \deg \Phi$) samples, see Fig. 8.1b. Furthermore, taking into account the periodicity of $\varphi(k)$ by using $\Phi = 1 - \mathrm{q}^{-30}$, the entire signal is eliminated after an initial transient of $30$ ($= \deg \Phi$) samples. □

## 8.2 Internal model principle—annihilation

The system

$$Ay = Bu + C\varphi$$

in closed-loop with the controller

$$Ru = -Sy + Tr$$

has the closed-loop dynamics

$$\begin{aligned} y &= \tfrac{BT}{A_c} r + \tfrac{RC}{A_c}\varphi \\ u &= \tfrac{AT}{A_c} r - \tfrac{SC}{A_c}\varphi \end{aligned}$$

Suppose that the disturbance $\varphi$ can be modeled as

$$\Phi(\mathrm{q}^{-1})\varphi(k) = 0, \quad k \geq \deg \Phi$$

The disturbance is eliminated asymptotically from the output $y$ if

$$\frac{RC}{A_c}\varphi(k) \to 0, \quad k \to \infty$$

This is true if the following two conditions are satisfied

**1.** $\Phi$ is a factor in $R$, i.e. $R = \Phi R_1$.

**2.** $A_c$ is stable ($|\lambda_i| < 1, \quad \lambda_i : A_c(\lambda_i^{-1}) = 0$).

Including the disturbance model, described by $\Phi$, into the controller polynomial $R$ is called *internal model principle*. Other appropriate notations are *annihilation* or *absorption principle*.

## 8.3 Tracking

The internal model principle can also be used for tracking problems when the reference signal $r(k)$ can be described by a model with an annihilation polynomial. The reference can be introduced to the controller in advance since it is known beforehand. In this way the time delay of the system can be eliminated. The controller becomes

$$Ru(k) = -Sy(k) + Tr(k+d)$$

To simplify notation, introduce the undelayed polynomial $B_d = \mathrm{q}^{\mathrm{d}} B = b_d + b_{d+1}\mathrm{q}^{-1} + \ldots$. The closed-loop system is then

$$y(k) = \frac{BT}{A_c} r(k+d) = \frac{B_d T}{A_c} r(k)$$

### Asymptotic tracking by annihilation

A naive approach to make $y(k)$ follow $r(k)$ is to select $T = A_c/B_d$ such that $y(k) = r(k)$. This is a bad strategy since $B$ usually is badly damped or unstable making the control signal badly damped or even unstable. A better approach is to use the internal model principle by taking into account the knowledge of the reference model. This leads to smoother control action and asymptotic tracking. Using the reference model, the tracking error is

$$e(k) = r(k) - y(k) = [1 - \frac{B_d T}{A_c}]r(k) = \frac{A_c - B_d T}{A_c}\frac{N}{\Phi}\delta(k)$$

which is eliminated asymptotically if $A_c$ is stable and the unstable polynomial $\Phi$ is canceled by the numerator $A_c - B_d T$. Sometimes, but not always, it is preferable to cancel $A_c$ with $T$ in order to speed up the response. For a general treatment we therefore factorize $T = T_1 A_{c1}$ and $A_c = A_{c1} A_m$. Then, calculating $T_1$ from the polynomial equation

$$B_d T_1 + \Phi M = A_m \quad \rightarrow T_1, M$$

the tracking error becomes

$$e(k) = \frac{MN}{A_m}\delta(k) \rightarrow 0, \quad k \rightarrow \infty \tag{8.3}$$

In order to get the shortest transient, the smallest degree $\deg M = \deg B_d - 1$ should be selected. The choice $A_m = 1$ gives the fastest tracking response where the tracking error attains zero in a finite number of samples. This so called dead-beat behavior is not possible with continuous-time design. However, fast response costs in large control action which is seen from

$$u(k) = \frac{AT_1}{A_m} r(k + d)$$

Thus, $A_m$ can moderate the control effort. This is important if $r$ changes discontinuously, for example a step change, in which case $A_m = A_c$ may be a good choice.

## 8.4 Robust tracking

The design of $T$ above is model based. In practice, a model is never perfect and only describes the real system approximately. The discrepancy between the real system and the model, the model mismatch, might therefore deteriorate the performance if it is not considered in the design. Using the annihilation approach both in the feedforward and the feedback design results in robust performance. The tracking error caused by model mismatch is treated as a disturbance and rejected asymptotically. In order to illustrate this, assume that the real plant can be described by the model

$$A^* y = B^* u$$

while the model used for design is as before

$$Ay = Bu + \varphi$$

The disturbance now includes model mismatch

$$\varphi = (A - A^*)y - (B - B^*)u$$

The controller with reference model and preview action is

$$\begin{aligned} Ru(k) &= -Sy(k) + Tr(k+d) \\ \Phi r(k) &= N\varepsilon(k), \quad \varepsilon(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \end{aligned}$$

Using $A_c^* = A^*R + B^*S$ the closed-loop dynamics are

$$\begin{aligned} y(k) &= \tfrac{B^*T}{A_c^*} r(k+d) \\ u(k) &= \tfrac{A^*T}{A_c^*} r(k+d) \end{aligned}$$

which can be used to express the disturbance as

$$\varphi(k) = \frac{C}{A_c^*} r(k) = \frac{CN}{A_c^* \Phi} \delta(k)$$

where $C = [(A - A^*)B^* - (B - B^*)A^*]\mathrm{q}^d T$. The tracking error is

$$\begin{aligned} e(k) &= r(k) - y(k) = r(k) - [\tfrac{B_d T}{A_c} r(k) + \tfrac{R}{A_c} \varphi(k)] \\ &= \tfrac{A_m - B_d T_1}{A_m} \tfrac{N}{\Phi} \delta(k) - \tfrac{RCN}{A_c A_c^* \Phi} \delta(k) \end{aligned}$$

In order to make each of the two terms asymptotically zero the polynomial equations below should be satisfied,

$$\begin{cases} A\Phi R_1 + BS = A_c & \rightarrow R_1, S \\ B_d T_1 + \Phi M = A_m & \rightarrow T_1, M \end{cases}$$

where $R = \Phi R_1$, $T = T_1 A_{c1}$ and $A_c = A_m A_{c1}$. This makes the tracking error

$$e(k) = \frac{MN}{A_m} \delta(k) + \frac{R_1 CN}{A_c A_c^*} \delta(k) \rightarrow 0, \quad k \rightarrow \infty \tag{8.4}$$

if $A_c^*$ is stable.

## 8.5 Examples

Reconsider the servo system in Chapter 4 with the integrator dynamics

$$y(k) = y(k-1) + u(k-1)$$

It will now be illustrated how to track a triangular-like and a sawtooth reference signal. In the first case, a robust repetitive control design is chosen where all repetitive disturbances (model mismatch included) are rejected. In the second case, a simple design is chosen where the feedback part only rejects the bias due to the non-calibrated power electronics and where the $T$ polynomial is designed to smooth the transients cause by the discontinuities of the reference.

**Example: 8.2**

The reference $r$ is a triangular-like signal with a period of $18$ samples. A robust repetitive design can be obtained by including the annihilation polynomial $\Phi(\mathrm{q}^{-1}) = 1 - \mathrm{q}^{-18}$ in the feedback polynomial $R$, i.e. selecting the fixed factor $R_f = \Phi$. The augmented plant with $A' = A\Phi$ is of high order and it is not a trivial task to find a robust pole placement. Distributing poles arbitrary, for example along the positive real axis, is unlikely to work. Care must be taken in the pole placement in order to bound the sensitivity functions $S_y$ and $S_u$ as much as possible. By using the interactive computer program Sysquake it was possible to bound both sensitivity functions below $2.5$. This was done by interactive pole placement. The resulting response is shown in Fig. 8.2. □

**Example: 8.3**

Now a sawtooth reference is considered. If a controller designed to track ramps is used, there will be transients every time a new ramp starts. The feedback design is here chosen to be as simple as possible. The fixed factor $R_f = 1 - \mathrm{q}^{-1}$ is chosen to eliminate biases from the non-calibrated power electronics. Closed-loop poles are chosen as in Chapter 4 at $0.7$ and $0.8$. In order to reduce the transients, the time delay of the system is eliminated by introducing the reference one sample in advance. Here $B = \mathrm{q}^{-1}$, $B_d = 1$ and $\Phi = (1 - \mathrm{q}^{-1})^2$. The polynomial equation for the calculation of $T$ is

$$B_d T + \Phi M = A_c \quad \rightarrow T, M$$

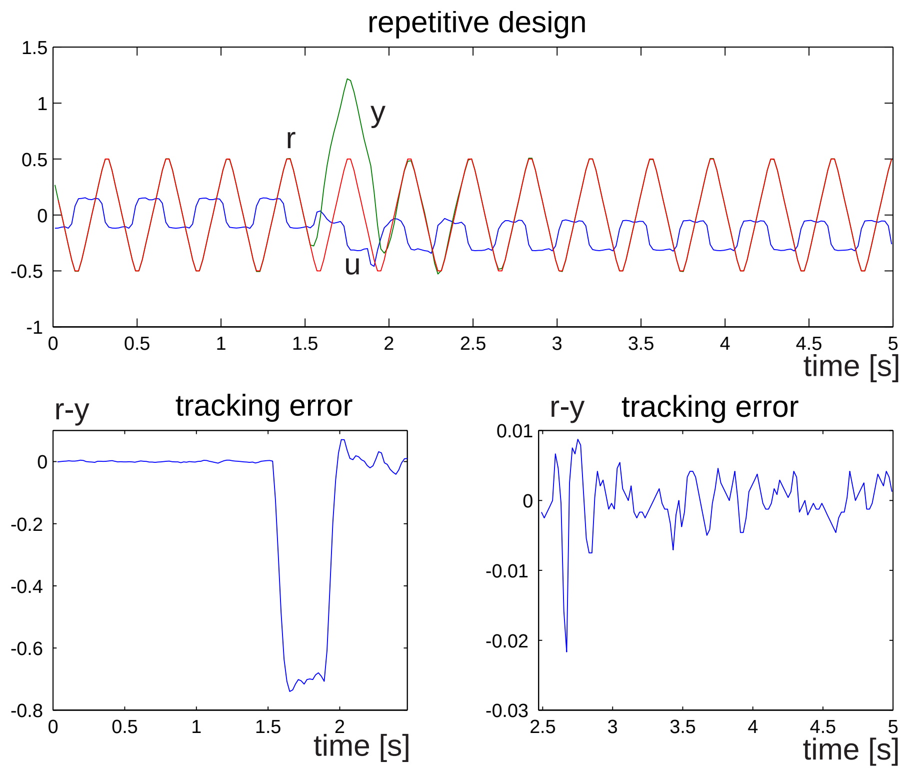



Figure 8.2: Repetitive tracking control. A step disturbance is introduced at $t = 1.5$.

One simple solution is $T = A_c$ and $M = 0$, which corresponds to cancellation of all dynamics. This gives rather large control actions. Another solution can be chosen, corresponding to the smallest $\deg T$, where less control effort is payed off by a transient ($M \neq 0$). The degree conditions are

$$\begin{cases} \deg T = \deg \Phi - 1 = 1 \\ \deg M = \max[\deg B_d - 1, \deg A_c - \deg \Phi] = \max[-1, 0] = 0 \end{cases}$$

The solution is found to be

$$\begin{cases} T = 0.44 - 0.38 \mathrm{q}^{-1} \\ M = 0.56 \end{cases}$$

The resulting response is shown in Fig. 8.3. Despite the one sample action in advance, there are overshoots. These can be reduced by acting further in advance.

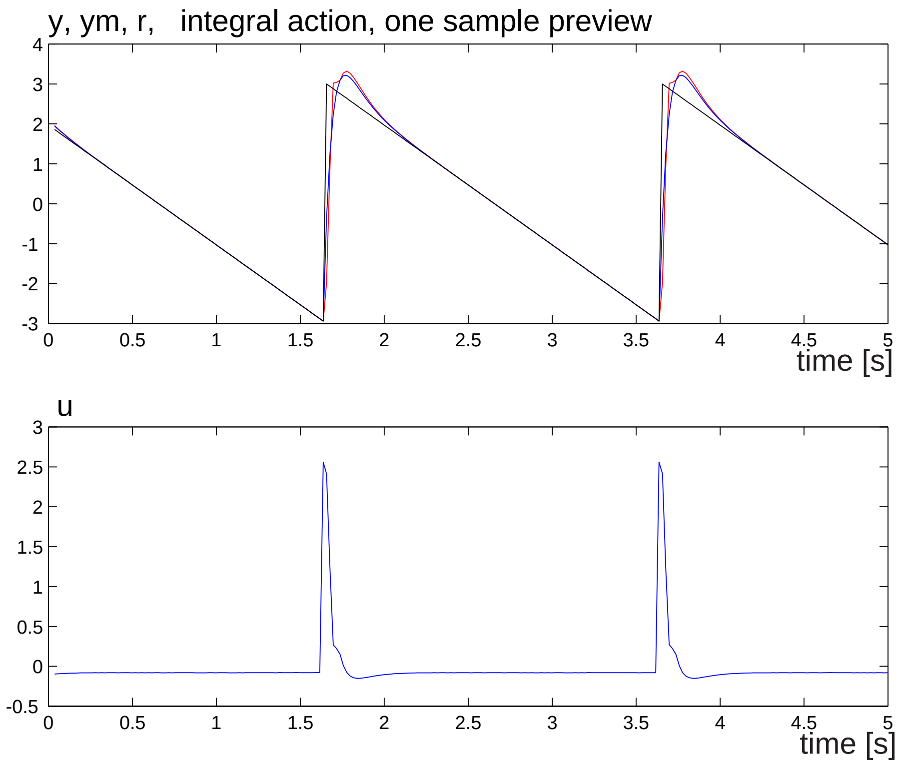



Figure 8.3: Action in advance by one sample.

Trying $3$ samples control action in advance corresponds to solving the polynomial equation

$$B_d T + \Phi M = \mathrm{q}^{-2} A_c \quad \rightarrow T, M$$

The degree conditions are

$$\begin{cases} \deg T = \deg \Phi - 1 = 1 \\ \deg M = \deg \mathrm{q}^{-2} A_c - \deg \Phi = 2 \end{cases}$$

and the solution is found to be

$$\begin{cases} T = 0.32 - 0.26\mathrm{q}^{-1} \\ M = -0.32 - 0.38\mathrm{q}^{-1} + 0.56\mathrm{q}^{-2} \end{cases}$$

The result is shown in Fig. 8.4. The overshoots have now disappeared and the control signal is smoother. □

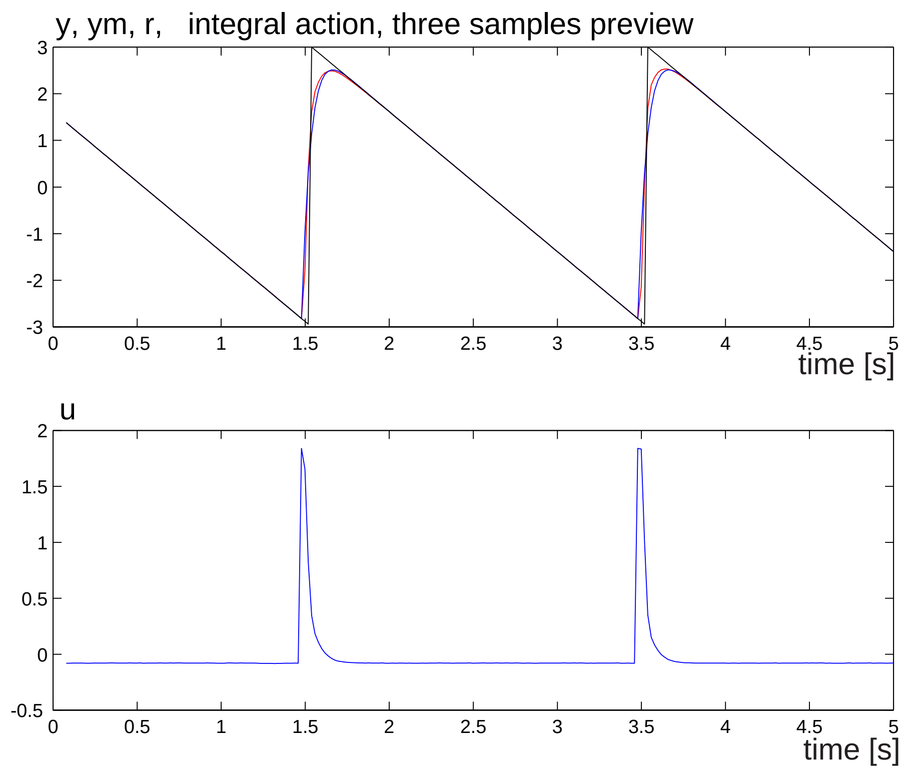



Figure 8.4: Action in advance by three samples.

# Exercises

Consider the system

$$y(k) = 0.5y(k-1) + u(k-2)$$

in the exercises 1-6 below.

**1.** Calculate the asymptotic error $e(\infty)$ ($e = r - y$) when the controller is $u(k) = Ke(k)$ (proportional controller) and the reference is

**(a)** $r(k) = 1$

**(b)** $r(k) = k$

**2.** Calculate the asymptotic error $e(\infty)$ when the controller is $u(k) = \frac{K}{1-\mathrm{q}^{-1}}e(k)$ (integral controller).

**3.** Let $r(k) = \sin 0.5k$. What is $\max|e(k)|$ (for large $k$) using the controller $u = Ke$ with $K = 0.5$ and $K = 2$?

**4.** Calculate a controller that gives the closed-loop characteristic polynomial $A_c = (1 - 0.4\mathrm{q}^{-1})(1 - 0.5\mathrm{q}^{-1})(1 - 0.6\mathrm{q}^{-1})$ and that makes the asymptotic error $e(\infty) = 0$ when $r(k) = 1$, despite modeling errors.

**5.** Use that the reference is known beforehand and find a controller that makes $e(k) = 0$ for all $r$ (step, ramp, sinusoid, etc.) and in addition achieves asymptotic tracking $e(\infty) = 0$ when a constant load disturbance is added to $y$.

**6.** Formulate the problem to solve for finding one fixed controller that eliminates the asymptotic error $e(\infty)$ for all of the three references: $r(k) = 1$, $r(k) = k$ and $r(k) = sin(\omega k)$, despite modeling errors.

## Solutions

The system in all exercises is described by $A = 1 - 0.5\mathrm{q}^{-1}$ and $B = \mathrm{q}^{-2}$. The error is $e(k) = \frac{AR}{A_c} r(k)$.

**1.** P-controller, i.e. $R = 1$ and $S = K$. Then $A_c = 1 - 0.5\mathrm{q}^{-1} + K\mathrm{q}^{-2}$. From the triangular stability region for a second order system it is concluded that the system is stable if and only if $-0.5 < K < 1$.

**a.** $r = 1$ Step: $e(k) = G(\mathrm{q}^{-1})r(k) \to G(1) \cdot 1$, $k \to \infty$, where $G = \frac{AR}{A_c}$. Thus, $e(\infty) = 0.5/(0.5 + K)$ if $-0.5 < K < 1$.

**b.** $r(k) = k$ Ramp: $e(k) = G(\mathrm{q}^{-1})r(k)$, where $G = (AR)/A_c$. Since $G(1) \neq 0$ then $e$ is a ramp, i.e. $e(k) \to \infty, \to \infty$.

**2.** I-controller: $R = 1 - \mathrm{q}^{-1}$ and $S = K$. Then $A_c = 1 - 1.5\mathrm{q}^{-1} + (0.5 + K)\mathrm{q}^{-2}$. Stability condition: $0 < K < 0.5$. Error $e(k) = Gr(k)$ with $G = \frac{AR}{A_c}$ and $G(1) = 0$. Therefore $e(\infty) = G(1)1 = 0$ when $r = 1$. If $r(k) = k$, then $r_1(k) = (1 - \mathrm{q}^{-1})r(k)$ is a step. In this case, ramp response to the system $G$ is the same as step response to the system $G/(1 - \mathrm{q}^{-1}) = G_1$. Thus, $e(\infty) = G_1(1)1 = 0.5/K$, if $0 < K < 0.5$.

**3.** P-controller: $e(k) = \frac{1-0.5\mathrm{q}^{-1}}{1-0.5\mathrm{q}^{-1}+K\mathrm{q}^{-2}} r(k) = G(\mathrm{q}^{-1})r(k)$. After a transient $e(k) \to |G(\mathrm{e}^{-\mathrm{i}0.5})| \sin(0.5k + \arg G(\mathrm{e}^{-\mathrm{i}0.5}))$, if $-0.5 < K < 1$. If $K =$

$0.5$,$\max|e(k)| = |G(\mathrm{e}^{-\mathrm{i}0.5})| = 0.72$. If $K = 2$ the system is unstable and $e(k) \to \infty$, $k \to \infty$.

**4.** Asymptotic tracking of a step despite modeling errors is only possible with integral action, i.e. $R_{fix} = 1 - \mathrm{q}^{-1}$ and $R = R_1 R_{fix}$. There is the factor $A$ in $A_c = A_{c1}A$. It therefore must also appear in $S = S_1 A$. Cancel it from the polynomial equation, which then becomes $R_{fix}R_1 + \mathrm{q}^{-2}S_1 = A_{c1}$. Find a solution with $\deg R_1 = \deg \mathrm{q}^{-2} - 1 = 1$ and $\deg S_1 = \deg R_{fix} - 1 = 0$. Thus, the equation is

$$(1 - \mathrm{q}^{-1})(1 + r_1\mathrm{q}^{-1}) + \mathrm{q}^{-2}s_0 = (1 - 0.4\mathrm{q}^{-1})(1 - 0.6\mathrm{q}^{-1})$$

Evaluate at $q = 1$ to get the equation: $s_0 = 0.6 \cdot 0.4 = 0.24$. Form the equation for the $\mathrm{q}^{-1}$-coefficients: $-1 + r_1 = -1 \Rightarrow r_1 = 0$. Controller $S/R = 0.24(1 - 0.5\mathrm{q}^{-1})/(1 - \mathrm{q}^{-1})$ and $T = S(1)$.

**5.** Asymptotic tracking for a constant (step) load disturbance $\Rightarrow R_{fix} = 1 - \mathrm{q}^{-1}$. Take the controller in 4 and adjust the $T$ polynomial to make $y(k) = r(k)$. Since $y(k) = \frac{BT}{A_c}r(k)$, and $B = \mathrm{q}^{-2}$ take $T = A_c\mathrm{q}^2$. This is 'noncausal' which means that the reference must be known in advance.

**6.** Absorption of steps and ramp by $(1 - \mathrm{q}^{-1})^2$ and absorption of sinusoids of frequency $\omega$ by $(1 - 2cos\omega\mathrm{q}^{-1} + \mathrm{q}^{-2})$. For absorption of all of these, choose $R_{fix} = (1 - \mathrm{q}^{-1})^2(1 - 2\cos\omega\mathrm{q}^{-1} + \mathrm{q}^{-2})$. Solve the polynomial equation $AR_{fix}R_1 + BS = A_c$ (stable) where $R = R_{fix}R_1$ and choose e.g. $T = S$.