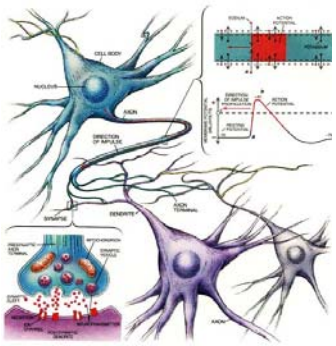
Artificial Intelligence

Statistical learning methods
Chapter 20, AIMA
(only ANNs & SVMs)

Artificial neural networks



The brain is a pretty intelligent system.

Can we "copy" it?

There are approx. 10^{11} neurons in the brain.

There are approx. 23×10^9 neurons in the male cortex (females have about 15% less).

The simple model

- The McCulloch-Pitts model (1943)

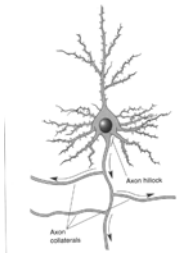
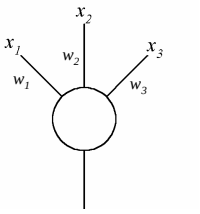
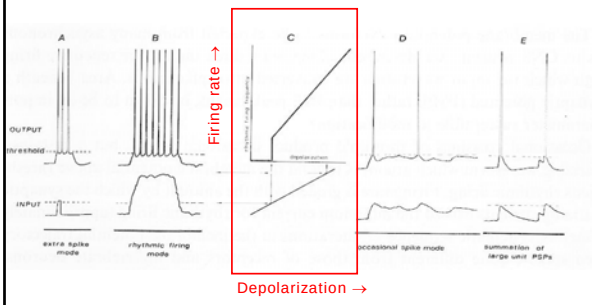


Image from
Neuroscience: Exploring the brain
by Bear, Connors, and Paradiso

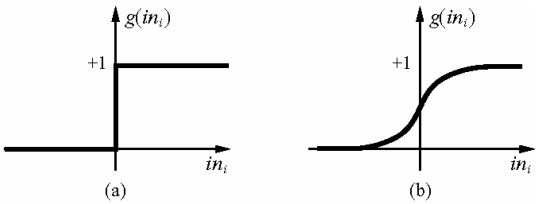


$$y = g(w_0 + w_1 x_1 + w_2 x_2 + w_3 x_3)$$

Neuron firing rate



Transfer functions *g(z)*



The Heaviside function

The logistic function

The simple perceptron

With {-1, +1} representation

$$y(\mathbf{x}) = \text{sgn}[\mathbf{w}^T\mathbf{x}] = \begin{cases} +1 & \text{if } \mathbf{w}^T\mathbf{x} > 0 \\ -1 & \text{if } \mathbf{w}^T\mathbf{x} < 0 \end{cases}$$

Traditionally (early 60:s) trained with *Perceptron learning*.

$$\mathbf{w}^T\mathbf{x} = w_0 + w_1 x_1 + w_2 x_2 + \cdots$$

Perceptron learning

Desired output $f(n) = \begin{cases} +1 & \text{if } \mathbf{x}(n) \text{ belongs to class } A \\ -1 & \text{if } \mathbf{x}(n) \text{ belongs to class } B \end{cases}$

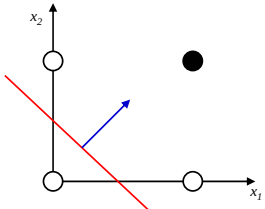
Repeat until no errors are made anymore

1. Pick a random example $[\mathbf{x}(n), f(n)]$
2. If the classification is correct, i.e. if $y(\mathbf{x}(n)) = f(n)$, then do nothing
3. If the classification is wrong, then do the following update to the parameters (η, the learning rate, is a small positive number)

$$w_i = w_i + \eta f(n) x_i(n)$$

Example: Perceptron learning

x_1	x_2	f
0	0	-1
0	1	-1
1	0	-1
1	1	+1



The AND function

Initial values:

$\eta = 0.3$

$\mathbf{w} = \begin{pmatrix} -0.5 \\ 1 \\ 1 \end{pmatrix}$

Example: Perceptron learning

x_1	x_2	f
0	0	-1
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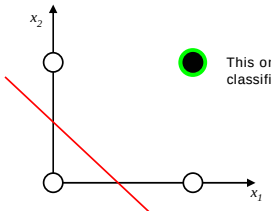

This one is correctly classified, no action.

The AND function

$\mathbf{w} = \begin{pmatrix} -0.5 \\ 1 \\ 1 \end{pmatrix}$

Example: Perceptron learning

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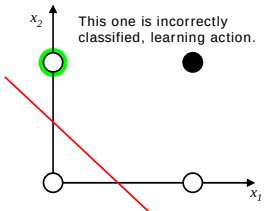

This one is incorrectly classified, learning action.

The AND function

$\mathbf{w} = \begin{pmatrix} -0.5 \\ 1 \\ 1 \end{pmatrix}$

$w_0 = w_0 - \eta \cdot 1 = -0.8$

$w_1 = w_1 - \eta \cdot 0 = +1$

$w_2 = w_2 - \eta \cdot 1 = 0.7$

Example: Perceptron learning

x_1	x_2	f
0	0	-1
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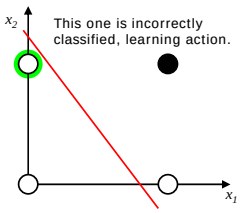

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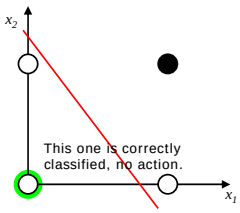

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The AND function

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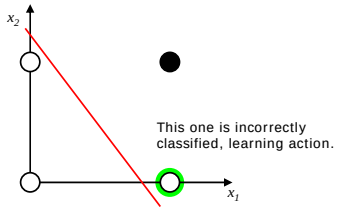

The AND function

$$\mathbf{w} = \begin{pmatrix} -0.8 \\ 1 \\ 0.7 \end{pmatrix}$$

$$w_0 = w_0 - \eta \cdot 1 = -1.1$$
$$w_1 = w_1 - \eta \cdot 1 = 0.7$$
$$w_2 = w_2 - \eta \cdot 0 = 0.7$$

Example: Perceptron learning

x_1	x_2	f
0	0	-1
0	1	-1
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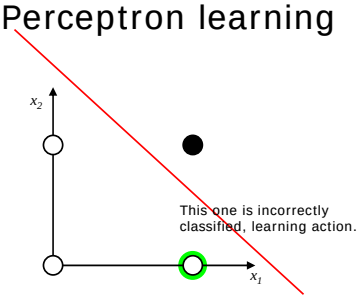
1	1	+1



The AND function

$$\mathbf{w} = \begin{pmatrix} -1.1 \\ 0.7 \\ 0.7 \end{pmatrix}$$

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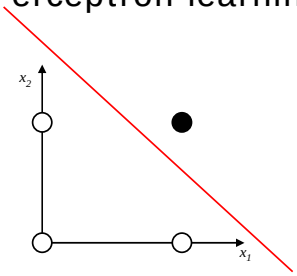


The AND function

$$\mathbf{w} = \begin{pmatrix} -1.1 \\ 0.7 \\ 0.7 \end{pmatrix}$$

Final solution

Perceptron learning

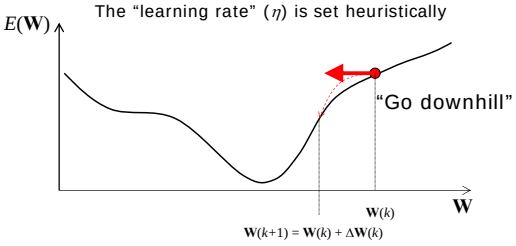
- Perceptron learning is guaranteed to find a solution in finite time, if a solution exists.
- Perceptron learning cannot be generalized to more complex networks.
- Better to use gradient descent – based on formulating an error and differentiable functions

$$E(\mathbf{W}) = \sum_{n=1}^{N} \left[f(n) - y(\mathbf{W}, n) \right]^2$$

Gradient search

$$\Delta \mathbf{W} = -\eta \nabla_W E(\mathbf{W})$$

The "learning rate" (η) is set heuristically

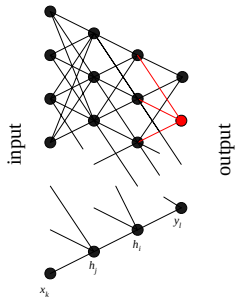


The Multilayer Perceptron (MLP)

- Combine several single layer perceptrons.
- Each single layer perceptron uses a sigmoid function (C^∞)
  E.g.

$$\phi(z) = \tanh(z)$$
$$\phi(z) = \left[1 + \exp(-z)\right]^{-1}$$

Can be trained using gradient descent



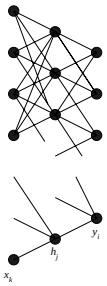
Example: One hidden layer

- Can approximate any continuous function

$$y_i(\mathbf{x}) = \theta\left[v_{i0} + \sum_{j=1}^{J} v_{ij} h_j(\mathbf{x}) \right]$$

$$h_j(\mathbf{x}) = \phi\left[w_{j0} + \sum_{k=1}^{D} w_{jk} x_k \right]$$

$\theta(z)$ = sigmoid or linear,
$\phi(z)$ = sigmoid.



Training: Backpropagation (Gradient descent)

$$\Delta \mathbf{w}(t) = -\eta \nabla_w E(t)$$

$$\Rightarrow$$

$$\Delta v_{ij}(t) = \eta \sum_{n=1}^{N} \delta_i(n,t) h_j[\mathbf{x}(n),t]$$

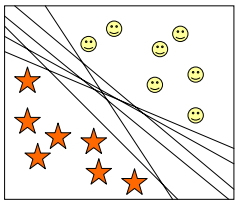
$$\Delta w_{jk}(t) = \eta \sum_{n=1}^{N} \delta_j(n,t) x_k(n)$$

where

$$\delta_j(n,t) = \sum_i \delta_i(n,t) v_{ij}(t) h_j'[\mathbf{x}(n),t]$$

Support vector machines

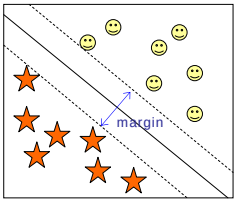
Linear classifier on a linearly separable problem



There are infinitely many lines that have zero training error.

Which line should we choose?

Linear classifier on a linearly separable problem



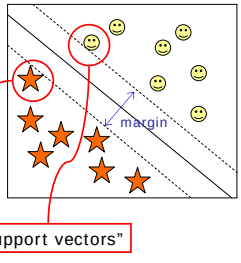
There are infinitely many lines that have zero training error.

Which line should we choose?

⇒ Choose the line with the largest margin.

The "large margin classifier"

Linear classifier on a linearly separable problem



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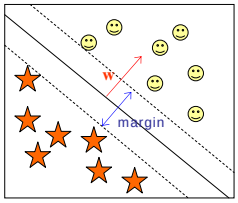
Computing the margin



The plane separating ★ and ☺ is defined by

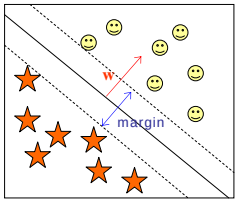
$$\mathbf{w}^T\mathbf{x} = a$$

The dashed planes are given by

$$\mathbf{w}^T\mathbf{x} = a + b$$

$$\mathbf{w}^T\mathbf{x} = a - b$$

Computing the margin



Divide by b

$$\mathbf{w}^T\mathbf{x}/b = a/b + 1$$

$$\mathbf{w}^T\mathbf{x}/b = a/b - 1$$

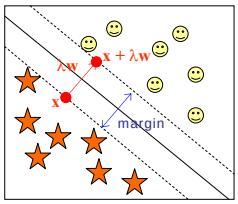
Define new $\mathbf{w} = \mathbf{w}/b$ and $\alpha = a/b$

$$\mathbf{w}^T\mathbf{x} = \alpha + 1$$

$$\mathbf{w}^T\mathbf{x} = \alpha - 1$$

We have defined a scale for $\mathbf{w}$ and b

Computing the margin



We have

$$\mathbf{w}^T\mathbf{x} = \alpha - 1$$

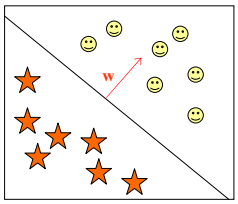
$$\mathbf{w}^T(\mathbf{x} + \lambda\mathbf{w}) = \alpha + 1$$

$$\|\lambda\mathbf{w}\| = \text{margin}$$

which gives

$$\text{margin} = \frac{2}{\|\mathbf{w}\|}$$

Linear classifier on a linearly separable problem



Maximizing the margin is equal to minimizing

$$\|\mathbf{w}\|$$

subject to the constraints

$\mathbf{w}^T\mathbf{x}(n) - \alpha \geq +1$ for all ☺

$\mathbf{w}^T\mathbf{x}(n) - \alpha \leq -1$ for all ★

Quadratic programming problem, constraints can be included with Lagrange multipliers.

Quadratic programming problem

Minimize cost (Lagrangian)

$$L_p = \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{n=1}^{N}\lambda_n\left[y(n)\left(\mathbf{w}^T\mathbf{x}(n) - \alpha\right) - 1\right]$$

Minimum of L_p occurs at the maximum of (the Wolfe dual)

$$L_D = \sum_{n=1}^{N}\lambda_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\lambda_n\lambda_m y(n)y(m)\underbrace{\mathbf{x}(n)^T\mathbf{x}(m)}$$

Only scalar product in cost. IMPORTANT!

Linear Support Vector Machine

Test phase, the predicted output

$$\hat{y}(\mathbf{x}) = \text{sgn}\left[\mathbf{w}^T\mathbf{x} - \alpha\right] = \text{sgn}\left[\sum_{n\in\Omega_s}\lambda_n y(n)\mathbf{x}(n)^T\mathbf{x} - \alpha\right]$$

Where α is determined e.g. by looking at one of the support vectors.

Still only scalar products in the expression.

How deal with nonlinear case?

- Project data into high-dimensional space $\mathbf{Z}$. There we know that it will be linearly separable (due to VC dimension of linear classifier).

$$\mathbf{z}(n) = \varphi[\mathbf{x}(n)]$$

- We don't even have to know the projection...!

Scalar product kernel trick

If we can find kernel such that

$$K(\mathbf{x}(n), \mathbf{x}(m)) = \boldsymbol{\varphi}(\mathbf{x}(n))^T \boldsymbol{\varphi}(\mathbf{x}(m))$$

Then we don't even have to know the mapping to solve the problem...

Valid kernels (Mercer's theorem)

Define the matrix

$$\mathbf{K} = \begin{pmatrix} K[\mathbf{x}(1),\mathbf{x}(1)] & K[\mathbf{x}(1),\mathbf{x}(2)] & \cdots & K[\mathbf{x}(1),\mathbf{x}(N)] \\ K[\mathbf{x}(2),\mathbf{x}(1)] & K[\mathbf{x}(2),\mathbf{x}(2)] & \cdots & K[\mathbf{x}(2),\mathbf{x}(N)] \\ \vdots & \vdots & \ddots & \vdots \\ K[\mathbf{x}(N),\mathbf{x}(1)] & K[\mathbf{x}(N),\mathbf{x}(2)] & \cdots & K[\mathbf{x}(N),\mathbf{x}(N)] \end{pmatrix}$$

If $\mathbf{K}$ is symmetric, $\mathbf{K} = \mathbf{K}^T$, and positive semi-definite, then $K[\mathbf{x}(i),\mathbf{x}(j)]$ is a valid kernel.

Examples of kernels

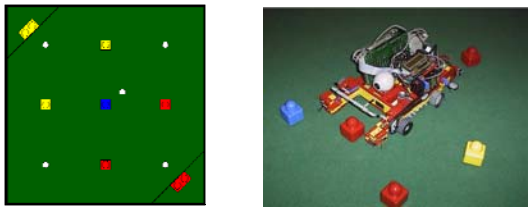
$$K[\mathbf{x}(i),\mathbf{x}(j)] = \exp\left[-\|\mathbf{x}(i)-\mathbf{x}(j)\|^2 / 2\sigma\right]$$
$$K[\mathbf{x}(i),\mathbf{x}(j)] = \left[\mathbf{x}(i)^T \mathbf{x}(j)\right]^d$$

First, Gaussian kernel.
Second, polynomial kernel. With d=1 we have linear SVM.

Linear SVM often used with good success on high dimensional data (e.g. text classification).

Example: Robot color vision

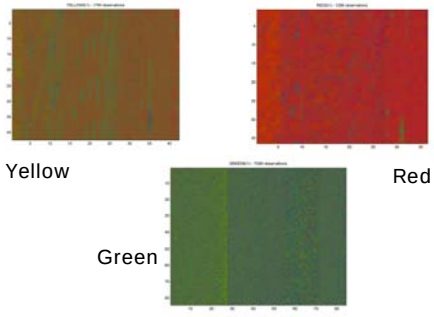
(Competition 1999)



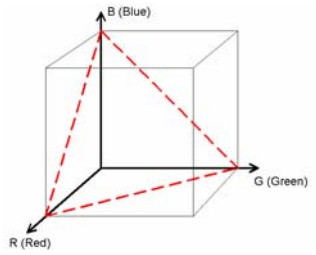
Classify the Lego pieces into *red*, *blue*, and *yellow*.
Classify *white* balls, *black* sideboard, and *green* carpet.

What the camera sees

(RGB space)



Mapping *RGB* (3D) to *rgb* (2D)

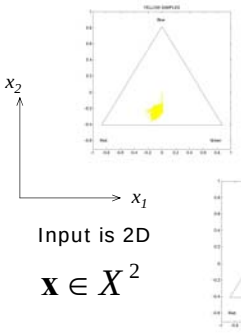


$$r = \frac{R}{R+G+B}$$

$$g = \frac{G}{R+G+B}$$

$$b = \frac{B}{R+G+B}$$

Lego in normalized *rgb* space



Input is 2D

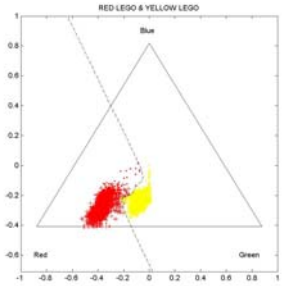
$\mathbf{x} \in X^2$

Output is 6D:
{red, blue, yellow, green, black, white}

$\mathbf{c} \in C^6$

MLP classifier

2-3-1 MLP
Levenberg-Marquardt

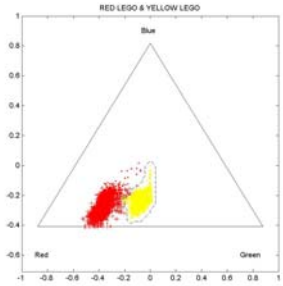


E_train = 0.21%
E_test = 0.24%

Training time
(150 epochs):
51 seconds

SVM classifier

SVM with
γ = 1000



E_train = 0.19%
E_test = 0.20%

Training time:
22 seconds

$$K(\mathbf{x},\mathbf{y}) = \exp\left[-\gamma(\mathbf{x}-\mathbf{y})^2\right]$$