Solutions to exam in Signals and Systems for IT-master, 051026

1. The total phase response goes from $0° \rightarrow -90°$ that means there should be a difference of first order between denominator and numerator.

There seems to be at least one zero in the transfer function due to the positive phase change and the decrease in the magnitude slope around 10 - 20 rad/s.

Low frequency gain : -20dB $(1/10) = K$

High frequency asymptote : -20 dB/dec.

"A plausible transfer function."

$$H(s) = \frac{K\left(\frac{s}{\omega_1}+1\right)}{\left(\frac{s}{\omega_2}+1\right)\left(\frac{s}{\omega_3}+1\right)}$$

Let see if we can find the values.

$K = 1/10$, the break frequencies will appear in the following order: ω_2, ω_1 and ω_3.

$\omega_2 \approx 1$ rad/sec , $\omega_1 \approx 10$ $\omega_3 = 100$ rad/s

Answer: $$\frac{\frac{1}{10}\left(\frac{s}{10}+1\right)}{\left(\frac{s}{1}+1\right)\left(\frac{s}{100}+1\right)}$$

2.

the output value depends only on present or past input and output values.

a) $$y[n] - \frac{1}{2} y[n-1] = x[n]$$

$$\rightarrow H(z) \; \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \leftarrow \text{poles } |z| < 1$$

b)

A causal, linear system and with memory could look like:

$$G(s) = \frac{1}{s+1} = \frac{Y(s)}{X(s)}$$

$$y(t) = -\frac{d}{dt} y(t) + x(t)$$

c) The system looks like: $y(t) = [\sin 6t]\, x(t)$

It's BIBO stable $\left| [\sin 6t] \right| \leq 1$, if $x(t)$ is bounded $\Rightarrow$ $y(t)$ is bounded.

There is no output that depends on future output / input. $\Rightarrow$ It's causal.

It lacks memory, if input is zero $\Rightarrow$ output will be zero.

It's time dependent $y(t_1) = (\sin 6t_1) \cdot x(t_1)$ will not be the same as $y(t_2) = (\sin 6t_2) \cdot x(t_2)$

It's linear!

3. Differential equation: $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = 2\frac{d}{dt}x(t)$

$$s^2 Y(s) + 5s\,Y(s) + 4Y(s) = 2s\,X(s)$$

Disregard the initial conditions, when we try to find the transfer function.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s^2+5s+4} \Rightarrow H(j\omega) = \frac{2j\omega}{5j\omega + 4 - \omega^2}$$

Magnitude response: $|H(j\omega)| = \frac{2\omega}{\sqrt{25\omega^2 + (4-\omega^2)^2}}$

Phase response:

$$\arg\{H(j\omega)\} = 90^\circ - \arctan\left(\frac{5\omega}{4-\omega^2}\right)\cdot\frac{180^\circ}{\pi}$$

The break frequencies from the transfer function are: 1, 4

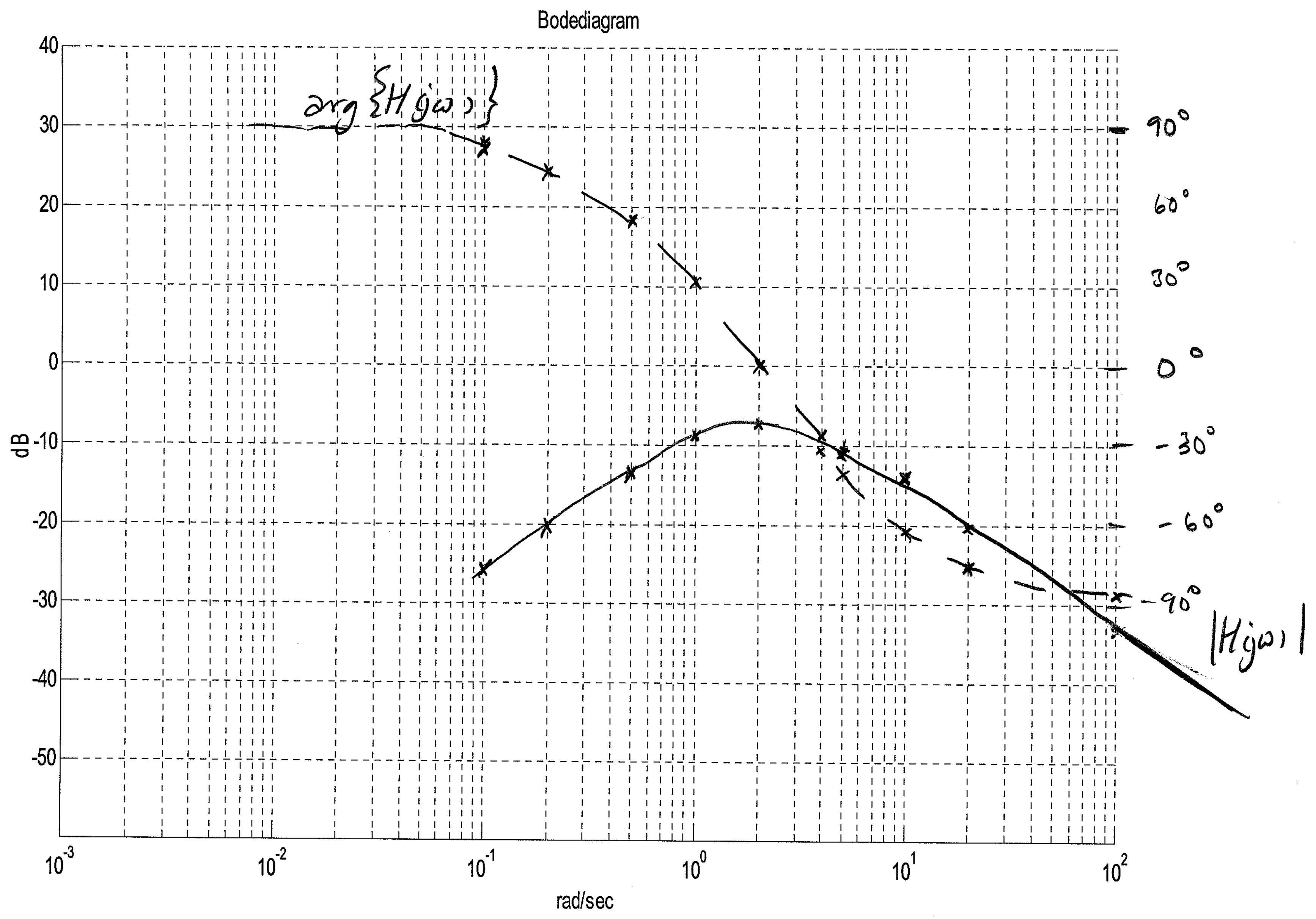
The most interesting happens around the break frequencies, so let focus on the decades around these two.

$$|H(j\omega)|_{dB} = 20\log^{10}\left(\frac{2\omega}{\sqrt{25\omega^2 + (4-\omega^2)^2}}\right)$$

ω	$\lvert H(j\omega)\rvert$	$\lvert H(j\omega)\rvert_{dB}$	$\arg\{H(j\omega)\}$
0.1	0.05	-26.1	82,8
0.2	0.098	-20.2	75,8
0.5	0.22	-13.1	56,3
1	0.34	-9.3	31
2	0,4	-8.0	0
4	0.34	-9.3	-31
5	0.31	-10.3	-40
10	0.18	-14.7	-62,5
20	0.098	-20.2	-75.8
100	0.02	-34	-87

Draw the Bode plot!

3.


Bodediagram
40
30
20
10
0
-10
-20
-30
-40
-50
dB
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
rad/sec
arg{H(jω)}
|H(jω)|
90°
60°
30°
0°
-30°
-60°
-90°

4.

a) To test whether a system is linear or not, we use the properties of superposition and homogeneity to decide.

- superposition:

$x_1(t) \rightarrow \boxed{H} \rightarrow y_1(t)$

$x_2(t) \rightarrow \boxed{H} \rightarrow y_2(t)$

$x_1(t) + x_2(t) \rightarrow \boxed{H} \rightarrow y_1(t) + y_2(t)$

- Homogeneity

$\xrightarrow{x_1(t)} \boxed{H} \rightarrow y_1(t)$

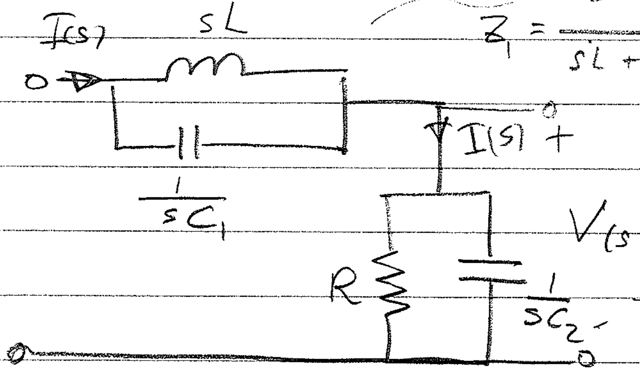
$\xrightarrow{a \cdot x_1(t)} \boxed{H} \rightarrow a \cdot y_1(t)$

b) See page 431 - 439 in the text book. The answer should involve the wave forms, spectrum, carrie wave

c)
- Easy to modify the filter charecteristics
- The filter will not face the same problems as an analog, like aging?
- good accuracy
- there is no limit for lower cutoff freq.

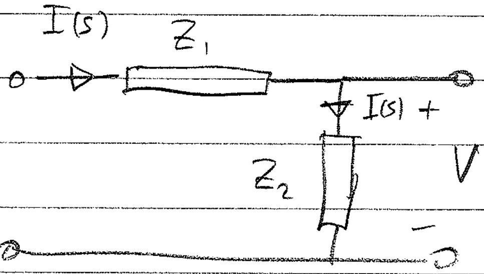
d)
- finite memory, low transient start-up.
- BIBO stable (always)
- linear phase response
- all poles in the origin.

5.



$$Z_1 = \frac{sL \cdot \frac{1}{sC_1}}{sL + \frac{1}{sC_1}} = \frac{sL}{s^2LC_1 + 1}$$

$$Z_2 = \frac{R \cdot \frac{1}{sC_2}}{R + \frac{1}{sC_2}} = \frac{R}{sRC_2 + 1}$$



Ohm's law gives:

$$V(s) = I(s) \cdot Z_2(s)$$

a)

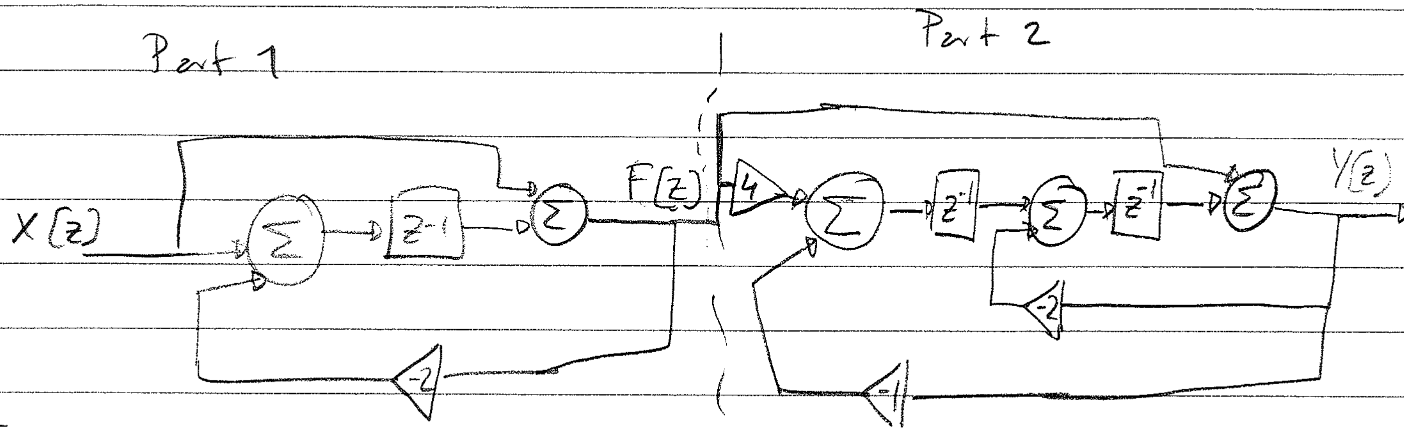
$$\left(H(s) = \right) Z(s) = \frac{V(s)}{I(s)} = \frac{R}{sRC_2 + 1} = \frac{500}{5 \cdot 10^{-3} s + 1}$$

b) We only have one pole $s = -200$
no zeros.

c) $z(t) = h(t) = 10^5 \cdot e^{-200t} \cdot u(t)$

d) The filter is of course a first order Low Pass filter.

6.



Part 1

$$\left(X[z] - 2F[z]\right) z^{-1} + X[z] = F[z]$$

$$X[z]\left(z^{-1}+1\right) = F[z]\cdot\left(1+2z^{-1}\right) \Rightarrow \boxed{F[z] = X[z]\cdot\frac{\left(z^{-1}+1\right)}{\left(1+2z^{-1}\right)}}$$

Part 2

$$\left(\left(4F[z] - Y[z]\right)z^{-1} - 2Y(z)\right)z^{-1} + F[z] = Y(z)$$

$$4F(z)\cdot z^{-2} - Y[z]z^{-2} - 2Y[z]\cdot z^{-1} + F[z] = Y(z)$$

$$\boxed{F[z] = \frac{Y[z]\left(1+z^{-2}+2z^{-1}\right)}{4z^{-2}+1}}$$

Eliminate the intermediate variable F[z]!

$$Y[z]\frac{\left(1+z^{-2}+2z^{-1}\right)}{4z^{-2}+1} = X[z]\cdot\frac{\left(z^{-1}+1\right)}{1+2z^{-1}}$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{\cancel{(1+z^{-1})}\cdot(1+4z^{-2})}{(1+2z^{-1})(1+z^{-1})^{\cancel{2}}} = \frac{1+4z^{-2}}{1+3z^{-1}+2z^{-2}}$$

$$y[n] = -3y[n-1] - 2y[n-2] + x[n] + 4x[n-2]$$

Impulse: $X[z] = 1 \Rightarrow y[n] = -3y[n-1] - 2y[n-2] + x[n] + 4x[n-2]$

Step: $X[z] = \frac{1}{1-z^{-1}}$

$$Y[z] = \frac{1}{1-z^{-1}}\cdot\frac{1}{1+2z^{-1}}\cdot\frac{1+4z^{-2}}{1+z^{-1}} = \frac{z\left(z^2+4\right)}{(z-1)(z+2)(z+1)} = z\left(\frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{z+1}\right)$$

Next page!

6. Continues

Partial fraction expansion gives:

$$A(z+2)(z+1) + B(z-1)(z+1) + C(z-1)(z+2) = z^2+4$$

$$\begin{array}{l} z^2: \\ z^1: \\ z^0: \end{array} \begin{cases} A+B+C = 1 \\ 3A + C = 0 \rightarrow C = -3A \\ 2A - B - 2C = 4 \end{cases} \Rightarrow \begin{cases} A + B - 3A = 1 \\ 2A - B + 6A = 4 \end{cases}$$

$$6A = 5$$

$$A = 5/6$$

$$\Rightarrow B = 1 + 10/6 = 16/6$$

$$C = -15/6$$

$$Y[z] = z\left(\frac{5/6}{z-1} + \frac{16/6}{z+2} - \frac{15/6}{z+1}\right) = \frac{5/6}{1-z^{-1}} + \frac{16/6}{1+2z^{-1}} + \frac{15/6}{1+z^{-1}}$$

$$y[n] = 5/6\ (1)^n \cdot u[n] + 16/6 \cdot (-2)^n \cdot u[n] + 15/6\ (-1)^n \cdot u[n]$$

7.

$$y[n] - 0{,}6\, y[n-1] = 0.4\, x[n], \quad x[n] = \sin(n\pi/2) \cdot u[n]$$

$$Y(z) = \frac{0.4\, X(z)}{1-0{,}6\, z^{-1}}$$

$$X[z] = \frac{z^{-1} \cdot \sin \pi/2}{1 - z^{-1} \cdot \underbrace{2 \cdot \cos \pi/2}_{=0} + z^{-2}} = \frac{z^{-1}}{1+z^{-2}}$$

$$Y[z] = \frac{0.4 \cdot z^{-1}}{(1-0.6z^{-1})(1+z^{-2})} = \frac{0{,}4\, z^2}{(z-0{,}6)(z^2+1)} =$$

$$= \frac{z(0{,}4z)}{(z-0{,}6)(z^2+1)} = z\left(\frac{A}{z-0{,}6} + \frac{Bz+C}{z^2+1}\right)$$

$$Az^2 + A + Bz^2 - 0{,}6Bz + Cz - 0{,}6C = 0{,}4z$$

$$\begin{array}{l} z^2: \\ z^1: \\ z^0: \end{array} \begin{cases} A+B = 0 \rightarrow A = -B \\ -0{,}6B + C = 0{,}4 \\ A - 0{,}6C = 0 \end{cases} \Rightarrow \begin{cases} -0{,}6B + C = 0{,}4 \\ -B - 0{,}6C = 0 \end{cases} \cdot(-0{,}6) \Leftrightarrow$$

$$1{,}36C = 0.4 \Rightarrow C = \frac{1}{3{,}4} = \frac{5}{17}$$

$$\Rightarrow A = \frac{3}{5} \cdot \frac{1}{3{,}4} = \frac{3}{17} \Rightarrow B = -3/17$$

7 continues

$$Y[z] = z\left(\frac{3/17}{(z-0{,}6)} + \frac{-3/17\, z + 5/17}{\underbrace{z^2+1}_{\text{poles on the unit circle}}} \right) =$$

$$= \frac{3/17}{1-0{,}6\, z^{-1}} + \frac{-3/17 + 5/17 \cdot z^{-1}}{1+z^{-2}}$$

$$y[n] = \underbrace{\frac{3}{17}(0{,}6)^{n-1} \cdot u[n-1]}_{\text{transient part}}$$

stationary part

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8. $y[n] - 1/3\, y[n-1] = x[n] - 1/2\, x[n-1]$

z-transform!

$$Y[z]\left(1 - 1/3\, z^{-1}\right) = X[z]\left(1 - \tfrac{1}{2} z^{-1}\right)$$

$$H(z) = \frac{Y[z]}{X[z]} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}} \Rightarrow H^{inv}(z) = \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$\boxed{H[z] \cdot H^{inv}[z] = 1}$$

the inverse of the filter becomes:

$$y[n] = \tfrac{1}{2} y[n-1] + x[n] - 1/3 \cdot x[n-1]$$

Assume $x[n]$ is a step, Calculate the step response!

n	$y[n] =$	$\frac{1}{2} y[n-1]$	$+ x[n]$	$-\frac{1}{3} x[n-1]$
0	1	0	1	0
1	14/12	1/2	1	$-\frac{1}{3}$
2	15/12	7/12	1	$-\frac{1}{3}$
3	31/24	15/24	1	$-\frac{1}{3}$
4	63/48	31/48	1	$-1/3$
5	127/96	63/96	1	$-1/3$
6	255/192	127/192	1	$-1/3$
7	511/384	255/384	1	$-1/3$