13. [J.M. Steele, 2001] Use the defintion of conditional expectation to prove the *tower property*: if $\mathcal{G}, \mathcal{H}$ are σ-algebras such that $X \in \mathcal{G} \subset \mathcal{H}$, then $E(E(X \mid \mathcal{H}) \mid \mathcal{G}) = E(X \mid \mathcal{G})$.

14. [J.M. Steele, 2001] Show that if $\mathcal{G} \subset \mathcal{H}$, $X \in \mathcal{G}$, $Y \in \mathcal{H}$, $E(|X|) < \infty$ and Y is bounded, then $E(XY \mid \mathcal{G}) = X\, E(Y \mid \mathcal{G})$.

15. Show that if $\{X_n\}$ is a supermartingale such that $E(X_n)$ is constant, then $\{X_n\}$ is a martingale.

16. Let $\{S_n : n \geq 1\}$ be a sequence which satisfies $S_n = \sum_{k=1}^{n} X_k$ where $X_1, \ldots, X_n$ are independent and

$$X_k = \begin{cases} 1 & \text{when } k = 1 \\ -k^2 & \text{w.p. } k^{-2} \text{ when } k = 2, 3, 4, \ldots \\ \frac{k^2}{k^2-1} & \text{w.p. } 1 - \frac{1}{k^2} \text{ when } k = 2, 3, 4, \ldots \end{cases}$$

   (a) Is $\{S_n\}$ a martingale, a submartingale or a supermartingale?

   (b) Calculate $\lim\limits_{n \to \infty} E(S_n)$.

17. Let $\{X_n\}$ be a random process with respect to the flow $\mathbb{F} = \{\mathcal{F}_n\} = \{\sigma(X_1, \ldots, X_n)\}$.

   (a) Show that $\{X_n\}$ is a martingale if $X_n = \sum_{k=1}^{n} \epsilon_k$ and $\{\epsilon_k\}$ is i.i.d.[2] with $\epsilon_k = \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$. What is this $\{X_n\}$ process called? What is the $\{\epsilon_k\}$ process called?

   (b) Let us assume that $\{X_n\}$ satisfies the condition $P(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_1 = x_1) = P(X_n = x_n \mid X_{n-1} = x_{n-1})$ and $E(X_n) = 0$ for all n. What is this $\{X_n\}$ process called? If $\{X_n\}$ is a martingale: prove this, if $\{X_n\}$ is not a martingale: give a counterexample.

   (c) Let $\{X_t : t \in \mathbb{R}^+\}$ be the process defined by $X_t = N_t - t$ where $\{N_t\}$ is a Poisson process with intensity 1. Now let $\{Y_n : n \in \mathbb{Z}^+\}$ be constructed from the process $\{X_t\}$ according to $Y_t = CX_t$, $C \in \mathbb{R}$ for all $t = 1, 2, 3, \ldots$ Prove that $\{Y_n\}$ is a martingale with respect to the flow $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

18. Let $\{S_n\}$ be the random walk $S_n = \sum_{i=1}^{n} \epsilon_i$ where $P(\epsilon_i = 1) = p$, $P(\epsilon_i = -1) = q$ and $p + q = 1$. Under what conditions is $\{S_n\}$ a

   (a) martingale

   (b) supermartingale

   (c) submartingale

   with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$.

[2] i.i.d. means *independent and identically distributed* and w.p. means *with probability*.

13 · Tower property $E(E(X|\mathcal{H})|\mathcal{G})$

[Breiman] where $\mathcal{G} \subset \mathcal{H}$

Let $G \in \mathcal{G}$. Then

$$E(E(X|\mathcal{H})|G) = E(E(X|\mathcal{H})\,I(G))$$

$$= \int_\Omega E(X|\mathcal{H})\,I(G)\,dP$$

$$= \int_G \underbrace{E(X|\mathcal{H})}_{\text{deterministic when integrating wrt } x}\,dP(x)$$

$$= \int_G x\,dP$$

$$= E(X|G)$$

Since this is true for all $G \in \mathcal{G}$ it holds that

$$E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{G})$$

14 $\mathcal{G} \subset \mathcal{H}$, $X \in \mathcal{G}$, $Y \in \mathcal{H}$, $E|X| < \infty$

Y bounded

$$\overset{\text{show}}{\Longrightarrow} \quad E(XY|\mathcal{G}) = X\, E(Y|\mathcal{G})$$

(why did we have to mention $\mathcal{H}$?)

Assume $G \in \mathcal{G}$. Then

$$E(XY|G) = E(XY\, I(G)) =$$

$$= \int_G x\, y\, dP(x,y)$$

$$\overset{X \in \mathcal{G}}{=} \int x\, y\, dP(y)$$

$$= x \int y\, dP(y)$$

$$= x\, E(Y|\mathcal{G})$$

Since true for all $G \in \mathcal{G}$

we have $E(XY|\mathcal{G}) = X\, E(Y|\mathcal{G})$

15 $E(X_{n+1} | \mathcal{F}_n) \leq X_n \wedge E(X_n) = c \Rightarrow E(X_{n+1}|\mathcal{F}_n) = X_n$

$$\begin{array}{c} c \\ \parallel \end{array} \quad E(X_{n+1} | f_n) = x_n + \varepsilon_n \text{ where } \varepsilon_n \geq 0$$

$E(X_{n+1}) = E(E(X_{n+1} | \mathcal{F}_n)) = E(X_n) = c \Rightarrow E(\varepsilon_n) = 0$. But $\varepsilon_n \geq 0$ so thus $\underline{\underline{\varepsilon_n = 0}} \Rightarrow E(X_{n+1}|f_n) = X_n$ Since true for any $f_n \subseteq \mathcal{F}_n$ we have

$$E(X_{n+1} | \mathcal{F}_n) = X_n$$

Alt. By Doob decomposition we can write a supermartingale X_n as

$X_n = M_n + A_n$ where M_n is a martingale an A_n is previsible wrt to $\mathcal{F}_n$. Thus $c = E(X_n) =$

$= E(M_n) + E(A_n) = d + E(A_n) \Rightarrow E(A_n)$ constant!

$E(X_{n+1} | \mathcal{F}_n) = E(M_{n+1} | \mathcal{F}_n) + E(A_{n+1} | \mathcal{F}_n) =$

$= M_n + A_{n+1} = X_n - A_n + A_{n+1}$

Constant $E(X_n)$ and constant $E(A_n)$ implies

$c = E(X_{n+1}) = E(E(X_{n+1} | \mathcal{F}_n)) =$

$= E(X_n - A_n + A_{n+1}) = c - E(A_n) + E(A_{n+1})$

$\Rightarrow E(A_n) = E(A_{n+1}) = 0 \Rightarrow X_n$ martingale

16 $S_n = \sum_{k=1}^{n} X_k$ $\quad X_k = \begin{cases} 1 & \text{if } k=1 \\ -k^2 & \text{w.p. } k^{-2} \text{ if } k \geq 2 \\ \frac{k^2}{k^2-1} & \text{w.p. } 1-k^{-2} \text{ if } k \geq 2 \end{cases}$

a) Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then

$$E(|S_n|) = E\left(\left|\sum_{k=1}^{n} X_k\right|\right) \leq \sum_{k=1}^{n} E(|X_k|) =$$

$$= 1 + \sum_{k=2}^{n} \left(|-k^2| \cdot k^{-2} + \left|\frac{k^2}{k^2-1}\right| \cdot \underbrace{(1-k^{-2})}_{=\frac{k^2-1}{k^2}} \right) =$$

$$= 1 + \sum_{k=2}^{n} (1+1) = 2n-1 < \infty$$

$$E(S_{n+1} \mid \mathcal{F}_n) = E\left(\sum_{k=1}^{n+1} X_k \mid X_1, \dots, X_n\right) =$$

$$= E\left(X_{n+1} + \sum_{k=1}^{n} X_k \mid X_1, \dots, X_n\right) =$$

$$= E(X_{n+1}) + S_n =$$

$$= (-n^2) \cdot n^{-2} + \frac{n^2}{n^2-1}(1-n^2) + S_n = -1+1+S_n = S_n$$

Thus $\{S_n\}$ is a martingale wrt $\{\mathcal{F}_n\}$.

b) $$\lim_{n\to\infty} E(S_n) = \lim_{n\to\infty} E\left(\sum_{k=1}^{n} X_k\right) =$$

$$= \sum_{k=1}^{\infty} E(X_k) = \sum_{k=1}^{\infty} (-1+1) = 0$$

$E(\lim_{n\to\infty} S_n) = ?$ (This is an exercise for the introduction course!)

17a) $X_n = \sum_{k=1}^{n} \varepsilon_k \qquad \{\varepsilon_k\}$ iid $= \begin{cases} -1 & \text{wp } \frac{1}{2} \\ 1 & \text{wp } \frac{1}{2} \end{cases}$

$$E(|X_n|) = E\left(\left|\sum_{k=1}^{n} \varepsilon_k\right|\right) \ \ldots \ < \infty$$

$$E(X_n \mid \mathcal{F}_{n-1}) = E(X_n \mid X_{n-1} = x_{n-1}, \ldots, X_1 = x_1)$$

$$= E\left(\sum_{k=1}^{n} \varepsilon_k \,\middle|\, \sum_{k=1}^{n-1} \varepsilon_k = x_{n-1}, \ldots, \varepsilon_1 = x_1\right)$$

$$= E\left(\sum_{k=1}^{n} \varepsilon_k - \sum_{k=1}^{n-1} \varepsilon_k + \sum_{k=1}^{n-1} \varepsilon_k \,\middle|\, \sum_{k=1}^{n-1} \varepsilon_k = x_{n-1}\right)$$

$$= E(\varepsilon_n) + E\left(\sum_{k=1}^{n-1} \varepsilon_k \,\middle|\, \sum_{k=1}^{n-1} \varepsilon_k = x_{n-1}\right)$$

$$= 0 \quad + \quad x_{n-1}$$

$\{X_n\}$ is a <u>random walk</u>

$\{\varepsilon_k\}$ is <u>(digital) noise</u>

If $\varepsilon_k \in N(0, \sigma^2)$ then <u>white noise</u>

17 b) Not a martingale!

Counterexample: let $\{X_n : n = 0, 1, 2, 3, \dots\}$ be defined by

$$X_n = \begin{cases} \begin{matrix} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{matrix} & \text{when } n = 0 \\ -X_{n-1} & \text{when } n = 1, 2, 3, \dots \end{cases}$$

Then obviously $E(X_n) = 0$ and $\{X_n\}$ is a Markov chain (show this!) but

$E(X_{n+1} \mid \mathcal{F}_n) = -X_n$.
immediately by the definition where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.
Thus $\{X_n\}$ is not a martingale.

17 c)

$X_t = N_t - t \quad \{N_t\} \text{ Poipr}(1)$

$Y_t = C X_t, \quad C \in \mathbb{R}, \quad t = 1, 2, 3, \ldots$

$\{Y_n\} \quad \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$

Let $Z_1 = N_1$, $Z_2 = N_2 - N_1$, $Z_3 = N_3 - N_2, \ldots$

Then $N_k = \sum_{j=1}^{k} Z_j$ and

$\{Z_j\}$ iid $Z_j \in \text{Poi}(1)$

Furthermore

$$Y_{k+1} = C(N_{k+1} - (k+1)) =$$

$$= C\left(\sum_{j=1}^{k+1} Z_j - k - 1\right)$$

$$= C\left(\sum_{j=1}^{k} Z_j - k + Z_{k+1} - 1\right)$$

$$= \underbrace{Y_k}_{\in \mathcal{F}_k} + C(Z_{k+1} - 1) \Rightarrow$$

$$\Rightarrow E(Y_{k+1} \mid \mathcal{F}_k) =$$

$$= E(Y_k + C(Z_{k+1} - 1) \mid \mathcal{F}_k)$$

$$= Y_k + C(E(\underbrace{Z_{k+1}}_{\in \text{Poi}(1)}) - 1)$$

$$= Y_k$$

18 $S_n = \sum_{i=1}^{n} \varepsilon_i$, $P(\varepsilon_i = 1) = 1 - P(\varepsilon_i = -1) = p$

(a) $E(|S_n|) \leq E \sum_{i=1}^{n} 1 = n < \infty$

Assume $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$. Then

$E(S_{n+1} | \mathcal{F}_n) = E(\varepsilon_{n+1} + S_n | \mathcal{F}_n) =$

$= E(\varepsilon_{n+1}) + S_n = p \cdot 1 + (1-p)(-1) + S_n =$

$= 2p - 1 + S_n$

Thus $\{S_n\}$ is a martingale iff $p = \frac{1}{2}$.

(b) $\{S_n\}$ is a supermartingale wrt $\{\mathcal{F}_n\}$
i.e. $E(S_{n+1} | \mathcal{F}_n) \leq S_n$ iff $p \leq \frac{1}{2}$

(c) $\{S_n\}$ is a submartingale wrt $\{\mathcal{F}_n\}$
i.e. $E(S_{n+1} | \mathcal{F}_n) \geq S_n$ iff $p \geq \frac{1}{2}$