Solutions to Exam for Random Processes, 7.5 ECTS

December 21, 2009, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume that the input signal $\{X_t : t \in \mathbb{R}\}$ with spectral density function $R_X(f)$ is filtered with impulse response h rendering the output $\{Y_t\}$. Prove that the spectral density of the output is $R_Y(f) = |H(f)|^2 R_X(f)$ where $H = \mathcal{F}(h)$. (4p)

   Solution: (See page 86 in the course literature.) □

2. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(1)$ process with $V(\epsilon_t) = 1$ and $V(X_t) = 2$. Calculate
   (a) $E(X_t)$, (3p)
   (b) the value of $|a_1|$. (3p)

   Solution:
   (a) $X_{t+1} = a_1 X_t + \epsilon_{t+1}$
   $\mu = E(X_{t+1}) = E(a_1 X_t + \epsilon_{t+1}) = a_1 E(X_t) + E(\epsilon_{t+1}) = a_1\mu + 0 \;\Rightarrow\; \mu = 0$.
   (b) $2 = V(X_{t+1}) = V(a_1 X_t + \epsilon_{t+1}) = a_1^2 V(X_t) + V(\epsilon_{t+1}) = a_1^2 \cdot 2 + 1 \;\Rightarrow\; a_1^2 = \frac{1}{2} \;\Rightarrow\; |a_1| = \frac{1}{\sqrt{2}}$. □

3. The process $\{X_t\}$ is shot noise with intensity 0.05 and pulse function

$$g(t) = \begin{cases} 2-t & \text{if } 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

   Determine the covariance function of $\{X_t\}$. (4p)

   Solution: According to the Campbell formulae we have $r(\tau) = \lambda \int g(u)g(u-\tau)\,du = 0.05 \int (2-u)I(0<u<2)\,(2-(u-\tau))I(0<u-\tau<2)\,du$ $\{\text{Assume } \tau > 0\}$ $= 0.05\int_\tau^2 (2-u)(2-u+\tau)\,du = 0.05\int_\tau^2 (4+2\tau-(4+\tau)u+u^2)\,du = 0.05[(4+2\tau)u - \frac{4+\tau}{2}u^2 + \frac{u^3}{3}]_\tau^2 = 0.05(2\cdot 4 - 2\cdot 4 + \frac{8}{3} + (4-2-4)\tau + (2-2)\tau^2 + (\frac{1}{2}-\frac{1}{3})\tau^3) = 0.05(\frac{8}{3} - 2\tau + \frac{1}{6}\tau^3)$. Since the cvf is symmetric $r(\tau) = 0.05(\frac{8}{3} - 2|\tau| + \frac{1}{6}|\tau|^3)$. □

4. A weakly stationary Gaussian process $\{X_t : t \in \mathbb{R}\}$ has expectation function $m_X = 1$ and covariance function $r_X(\tau) = \frac{1}{1+(\pi\tau)^2}$.

(a) Calculate $E(\int_0^1 t^{0.5} X_t\, dt)$. (3p)

(b) Calculate $P(X_t < 0.5X_{t+0.5})$. (4p)

(c) Assume $\{X_t\}$ is sampled in time-points $t = 0, \pm 1, \pm 2, \pm 3, \ldots$ Calculate the spectral density of the sampled signal. (4p)

Solution:

(a) $E(\int_0^1 t^{0.5} X_t\, dt) = \int_0^1 t^{0.5} E(X_t)\, dt = \int_0^1 t^{0.5} dt = [\frac{2}{3} t^{3/2}]_0^1 = \frac{2}{3}$.

(b) $P(X_t < 0.5X_{t+0.5}) = P(X_t - 0.5X_{t+0.5} < 0)$. Since $\{X_t\}$ is Gaussian, $X_t - 0.5X_{t+0.5} \in N(\mu, \sigma^2)$ where $\mu = E(X_t - 0.5X_{t+0.5}) = E(X_t) - 0.5E(X_{t+0.5}) = 1 - 0.5 \cdot 1 = 0.5$ and $\sigma^2 = V(X_t - 0.5X_{t+0.5}) = C(X_t - 0.5X_{t+0.5}, X_t - 0.5X_{t+0.5}) = V(X_t) + 2 \cdot (-0.5) C(X_t, X_{t+0.5}) + (-0.5)^2 V(X_{t+0.5})$. According to the cvf $V(X_t) = V(X_{t+0.5}) = \frac{1}{1+0^2} = 1$ and $C(X_t, X_{t+0.5}) = \frac{1}{1+(\pi \cdot 0.5)^2} = \frac{4}{4+\pi^2}$. Thus $\sigma^2 = 1 - \frac{4}{4+\pi^2} + 0.25 = 0.9616$ and $P(X_t - 0.5X_{t+0.5} < 0) = \Phi(\frac{0-0.5}{\sqrt{0.9616}}) = 1 - \Phi(0.5099) = 0.305$.

(c) According to the tables, if $G(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$, then we have with $\alpha = 2$ that $G(f) = \frac{1}{1+\pi^2 f^2}$ implying that $g(\tau) = e^{-2|\tau|}$. Due to the inversion theorem, for $r_X(\tau) = \frac{1}{1+(\pi\tau)^2}$ we therefore get $R_X(f) = e^{-2|f|}$. Now from the summary of formulae with $d = 1$ we have that $f \in (-\frac{1}{2}, \frac{1}{2}]$ and $R_Y(f) = \sum_{k=-\infty}^{\infty} R_X(f+k) = \sum_{k=-\infty}^{\infty} e^{-2|f+k|} = \sum_{k=-\infty}^{-1} e^{2(f+k)} + e^{-2|f|} + \sum_{k=1}^{\infty} e^{-2(f+k)} = (e^{2f} + e^{-2f}) \sum_{k=1}^{\infty} e^{-2k} + e^{-2|f|} = (e^{2f} + e^{-2f}) \frac{e^{-2}}{1-e^{-2}} + e^{-2|f|} = \frac{2\cosh(2f)}{e^2+1} + e^{-2|f|}$. □

5. Let the process $X = \{X_t : t \in \mathbb{R}^+\}$ be defined by $X_t = (-1)^{A+N_t}$ where $N = \{N_t : t \in \mathbb{R}^+\}$ is a Poisson process with intensity $\lambda = 1$, and A is a random variable such that $P(A = 0) = P(A = 1) = \frac{1}{2}$ independently of N. Prove that X is weakly stationary. (5p)

Solution: To prove weak stationarity we should check that $E(X_t) = m$ and $C(X_s, X_t) = r(|s - t|)$ for all $s, t \in \mathbb{R}^+$. Since A and N are independent we have $E(X_t) = E((-1)^{A+N_t}) = E((-1)^A) E((-1)^{N_t}) = 0$ since $E((-1)^A) = (-1)^0 P(A = 0) + (-1)^1 P(A = 1) = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$. Further $C(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t) = E((-1)^A (-1)^{N_s} (-1)^A (-1)^{N_t}) = E((-1)^{2A}) E((-1)^{N_s + N_t}) \overset{s \le t}{=} E((-1)^{N_s + N_s + (N_t - N_s)}) = E((-1)^{2N_s}) E((-1)^{N_t - N_s}) = E((-1)^{N_t - N_s})$ where $N_t - N_s \in Poi(\lambda(t-s))$. Now, since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$ (also negative x) we have that $E((-1)^{N_t - N_s}) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{(t-s)^k}{k!} e^{-(t-s)} = e^{-(t-s)} \sum_{k=0}^{\infty} \frac{(-(t-s))^k}{k!} = e^{-(t-s)} e^{-(t-s)} \sum_{k=0}^{\infty} \frac{(-(t-s))^k}{k!} e^{t-s} = e^{-2(t-s)}$. When $s \ge t$ we get correspondingly $E((-1)^{N_t - N_s}) = e^{-2(s-t)}$. Thus, in general $E((-1)^{N_t - N_s}) = e^{-2|s-t|}$ and finally $C(X_s, X_t) = e^{-2|s-t|}$ which is a function only of the time distance $|s - t|$, not of the location s and t. Therefore $\{X_t\}$ is weakly stationary. □