Exam for Random Processes, 7.5 ECTS

December 19, 2003, 9.00 am − 1.00 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

For each problem a *complete* solution should be given.
All solutions should be thoroughly presented.
Each soloution should start at the top of a new sheet of paper.
Only one solution a sheet.
The proper solutions will be available on internet at
`http://www.hh.se/staff/erja` → `teaching` → `random processes` → `031219: solution`

1. Show that <u>if</u> $\{X_t\}$ is an $AR(p)$ process, <u>then</u> its spectral density function is

$$R_X(f) = \frac{\sigma_\epsilon^2}{|\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2} \qquad \text{(3p)}$$

2. Show that <u>if</u> $\{X_t\}$ is weakly stationary and differentiable, <u>then</u> $E(X_t') = 0$. (3p)

3. Let $X = \begin{cases} 0 & \text{w.p. } 1/3 \\ 1 & \text{w.p. } 2/3 \end{cases}$ and independently $\{Y_t\}$ be a Poisson process with intensity 0.05. Then what is
   (a) $P(X > Y_3)$? (2p)
   (b) $E((-1)^{X+Y_{40}})$? (Hint: $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda$ and $\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \frac{1}{2}(e^\lambda + e^{-\lambda})$) (3p)

4. Let $\{X_t\}$ be defined by the relation $X_t = \epsilon_t - \epsilon_{t-1} + 2\epsilon_{t-2}$ for all $t \in \mathbb{Z}$ where $\{\epsilon_t\}$ is a sequence of independent variables all distributed $N(1,1)$. Calculate $P(|\frac{X_t + X_{t-1} - X_{t-2}}{3}| < 2)$. (3p)

5. A signal $\{X_t : t \in \mathbb{R}\}$ has cvf $r(\tau) = \frac{1}{1+\tau^2}$. The cost of sampling from $\{X_t\}$ is described by $C(d) = e^{\pi/d}(2\pi d + \frac{1}{d})$ where d is the lenght of the sampling interval. What is the minimal value of the product, $Err(d)C(d)$, of error proportion due to the alias effect, $Err(d)$, and sampling cost, $C(d)$? (4p)

6. Let $\{W_t\}$ be a standard Wiener process (i.e. a Wiener process with $\sigma^2 = 1$) and let $X_t = e^{-t/2} W_{e^t}$ for all $t \in \mathbb{R}$. Then $\{X_t\}$ is a *Gauss-Markov process.*
   (a) Calculate expectation and covariance function of $\{X_t\}$. (3p)
   (b) Show that $\{X_t\}$ is strongly stationary. (3p)

   Let $Y_t = \int_{t-1}^{t+1} X_u \, du$ for all $t \in \mathbb{R}$. Calculate
   (c) the spectral density function of $\{Y_t\}$. (4p)
   (d) the probability $P(X_t < Y_t)$. (2p)

GOOD LUCK!