Solutions to the Exam for Random Processes, 5p

January 18, 2003, 9.00 am − 13.00 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
Allowed aids:
Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that if $G = \mathcal{F}(g)$ and $h(\tau) \equiv G(\tau)$ then är $H(f) \equiv g(-f)$ (where $H = \mathcal{F}(h)$ and $g = \mathcal{F}^{-1}(G)$). (3p)

   Solution: Assume that $G = \mathcal{F}(g)$. Then

   $$g(-\tau) = \mathcal{F}^{-1}(G)(-\tau) = \int_{\mathbb{R}} e^{i2\pi f\cdot(-\tau)} G(f)\,df = \int_{\mathbb{R}} e^{-i2\pi f\tau} G(f)\,df$$

   i.e.

   $$g(-f) = \int_{\mathbb{R}} e^{-i2\pi f\tau} G(\tau)\,d\tau = \int_{\mathbb{R}} e^{-i2\pi f\tau} h(\tau)\,d\tau = \mathcal{F}(h)(f) = H(f)$$

   $\square$

2. Show that if $\{X_t : t \in \mathbb{R}\}$ is weakly stationary and $Z_t = X_t$ för $t \in \{dk : k \in \mathbb{Z}\}$ then $S_Z(f) = \sum_{-\infty}^{\infty} S_X(f + \frac{k}{d})$ for $-\frac{1}{2d} < f < \frac{1}{2d}$. (4p)

   Solution: (Se Lindgren-Rotzén, pp 75 − 76.)

3. Assume that $\{X_t\}$ och $\{Y_t\}$ are mutually independent Poisson processes with parameter $\lambda_X = 0.1$ and $\lambda_Y = 0.2$ respectively.
   (a) Calculate $P(X_{200} < Y_{101})$ approximatively. (3p)
   (b) Show that the process $Z_t = X_{2t} - Y_t$ is not stationary. (3p)

   Solution:
   (a) $X_t = \sum_{k=1}^{t} Z_k$ where $Z_1, Z_2, \dots, Z_t$ are independent and $Z_k \in Po(0.1)$.
   According to C.L.T. this means that approximately $X_t \in N(0.1t, \sqrt{0.1t})$.
   $Y_t = \sum_{k=1}^{t} W_k$ where $W_1, W_2, \dots, W_t$ are independent and $W_k \in Po(0.2)$.
   According to C.L.T. this means that approximately $Y_t \in N(0.2t, \sqrt{0.2t})$.
   Further $\mathsf{E}(X_{200} - Y_{101}) = \mathsf{E}(X_{200}) - \mathsf{E}(Y_{101}) = 0.1 \cdot 200 - 0.2 \cdot 101 = -0.2$
   and, since $\{X_t\}$ and $\{Y_t\}$ are independent, we have that
   $\mathsf{V}(X_{200} - Y_{101}) = \mathsf{V}(X_{200}) + \mathsf{V}(Y_{101}) = 0.1 \cdot 200 + 0.2 \cdot 101 = 40.2$
   and hence $X_{200} - Y_{101} \in N(-0.2, \sqrt{40.2})$ so
   $\mathsf{P}(X_{200} < Y_{101}) = P(X_{200} - Y_{101} < 0) = \Phi(\frac{0-(-0.2)}{\sqrt{40.2}}) = \Phi(0.0315) = \underline{0.512}$

(b) $Z_t = X_{2t} - Y_t$. For weak stationarity of $\{Z_t\}$ it is necessary that $\mathsf{E}(Z_t) = m$ is not a function of t and that $\mathsf{C}(Z_s, Z_t) = r_Z(s-t)$.
If the process $\{Z_t\}$ is weakly stationary this does not imply strict stationarity *but* if it is *not* weakly stationary then it cannot be strictly stationary, i.e. then it can't be stationary in any sense. We have that $\mathsf{E}(Z_t) = \mathsf{E}(X_{2t} - Y_t) = \mathsf{E}(X_{2t}) - \mathsf{E}(Y_t) = 0.1 \cdot 2t - 0.2 \cdot t = 0$ (ok, it might still be weakly stationary...)
Since $\{X_t\}$ and $\{Y_t\}$ are Poisson processes,
$\mathsf{C}(X_s, X_t) = 0.1 \min(s,t)$ and $\mathsf{C}(Y_s, Y_t) = 0.2 \min(s,t)$. Therefore
$\mathsf{C}(Z_s, Z_t) = \mathsf{C}(X_{2s} - Y_s\ ,\ X_{2t} - Y_t) =$
$= \mathsf{C}(X_{2s}, X_{2t}) - \mathsf{C}(X_{2s}, Y_t) - \mathsf{C}(Y_s, X_{2t}) + \mathsf{C}(Y_s, Y_t) =$
$= 0.1 \min(2s, 2t) - 0 - 0 + 0.2 \min(s,t) = 0.4 \min(s,t)$
However, $\min(s,t)$ is not a function of $s-t$ so $\{Z_t\}$ is not weakly stationary and thus $\{Z_t\}$ is neither weakly nor strictly stationary! □

4. The spectral density of the process $\{X_t\}$ is $S_X(f) = |f|e^{-af^2}$. This signal will be sampled from with interval 0.3. What does a have to be for not risking that more than 10% of the signal is distorted? (3p)

Solution: Approximately 4 "tailparts" are distorted so

$$4\int_{1/(2\cdot 0.3)}^{\infty} R_X(f)\,df \;\le\; 0.1\int_{-\infty}^{\infty} R_X(f)\,df$$

$\int_{1/0.6}^{\infty} |f|e^{-af^2}df = \frac{1}{2a}\int_{1/0.6}^{\infty} 2afe^{-af^2}df = \left\{ \begin{array}{ll} u = af^2 & f = 1/0.6 \\ du = 2af\,df & u = a(1/0.6)^2 = a/0.36 \end{array} \right\} =$
$= \frac{1}{2a}\int_{a/0.36}^{\infty} e^{-u}du = \frac{1}{2a}[-e^{-u}]_{a/0.36}^{\infty} = \frac{1}{2a}(0 - (-e^{-a/0.36})) = e^{-a/0.36}/(2a)$

$\int_{-\infty}^{\infty} |f|e^{-af^2}df = 2\int_0^{\infty} fe^{-af^2}df = 2 \cdot \frac{1}{2a}\int_0^{\infty} e^{-u}du = \frac{1}{a}(0 - (-1)) = 1/a.$

Therefore a must satisfy the condition

$$4 \cdot \frac{e^{-a/0.36}}{2a} \;\le\; 0.1 \cdot \frac{1}{a}$$

$\Rightarrow\quad 2e^{-a/0.36} \le 0.1$
$\Rightarrow\quad -a/0.36 \le \ln(0.1/2)$
$\Rightarrow\quad a \ge -0.36 \ln 0.05 = 1.078$ (Obs! Multiplying negative numbers switches the inequality.)

Answer: a must be *larger* than 1.078. □

5. Assume that $\{X_t\}$ is an $AR(1)$ process with regression parameter $a_1 = -2/3$ and noise process $\{\epsilon_t\}$ where $\epsilon_t \in N(0, 1/2)$.

   (a) What does $\mathsf{E}(X_t)$ and $\mathsf{V}(X_t)$ have to be for the process $\{X_t\}$ to be strictly stationary? (3p)

   (b) Calculate expectation function and covariance function of the derivative process $\{X_t'\}$. (3p)

Solution: $\{X_t\}$ AR(1) with $a_1 = -\frac{2}{3}$ means that $X_t - \frac{2}{3}X_{t-1} = \epsilon_t$ i.e. that $X_t = \frac{2}{3}X_{t-1} + \epsilon_t$ where $\epsilon_t \in N(0, \frac{1}{2})$.

(a) By definition all AR-processes are weakly stationary. Since the noise is normally distributed the process is a normal process and therefore weak stationarity implies strict stationarity.

Weak stationarity implies that $\mathsf{E}(X_t)$ and $\mathsf{V}(X_t)$ are constant with respect to t. Since $X_t = \frac{2}{3}X_{t-1} + \epsilon_t$ we have, by taking the expectation of both sides of this equality, that
$m = \mathsf{E}(X_t) = \mathsf{E}(\frac{2}{3}X_{t-1} + \epsilon_t) = \frac{2}{3}\mathsf{E}(X_t) + \mathsf{E}(\epsilon_t) = \frac{2}{3}m + 0 \Rightarrow$
$\Rightarrow m - \frac{2}{3}m = 0 \Rightarrow m = 0$ and by taking the variance of both sides we have that $\sigma^2 = \mathsf{V}(X_t) = \mathsf{V}(\frac{2}{3}X_{t-1} + \epsilon_t) = \frac{4}{9}\mathsf{V}(X_{t-1}) + \mathsf{V}(\epsilon_t) = \frac{4}{9}\sigma^2 + \frac{1}{4} \Rightarrow$
$\Rightarrow \sigma^2(1 - \frac{4}{9}) = \frac{1}{4} \Rightarrow \sigma^2 = \frac{1}{4} \cdot \frac{9}{5} = \frac{9}{20} = 0.45$

Answer: $\underline{\mathsf{E}(X_t) = 0 \text{ and } \mathsf{V}(X_t) = 0.45}$

(b) According to Thm (p 127, Lindgren-Rotzén) we have that
<u>if</u> $\{X_t\}$ is weakly stationary
<u>then</u> $m_{X'} = 0$ and $r_{X'}(\tau) = r''_X(\tau)$.

The expectation function is thus constantly 0.
Now we may calculate $R_X(\tau) = \mathsf{C}(X_t, X_{t+\tau})$ and differentiate this twice and thereby get $R_{X'}(\tau)$ according to the theorem. We have that
$\mathsf{C}(X_t, X_{t+1}) = \mathsf{C}(X_t, \frac{2}{3}X_t + \epsilon_{t+1}) = \frac{2}{3}\mathsf{V}(X_t) + 0 = \frac{2}{3} \cdot \frac{9}{20}$
$\mathsf{C}(X_t, X_{t+2}) = \mathsf{C}(X_t, \frac{2}{3}X_{t+1} + \epsilon_{t+2}) = \frac{2}{3}\mathsf{C}(X_t, X_{t+1}) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{9}{20} = (\frac{2}{3})^2 \frac{9}{20}$
$\vdots$
$\mathsf{C}(X_t, X_{t+\tau}) = (\frac{2}{3})^\tau \frac{9}{20} = r_X(\tau)$. Thus
$r'_X(\tau) = D(e^{\tau \ln(2/3)} \frac{9}{20}) = \frac{9}{20} \ln \frac{2}{3} e^{\tau \ln(2/3)} = \frac{9}{20} \ln \frac{2}{3} (\frac{2}{3})^\tau$
$r''_X(\tau) = \frac{9}{20} \ln \frac{2}{3} \ln \frac{2}{3} (\frac{2}{3})^\tau = \frac{9}{20} (\ln \frac{2}{3})^2 (\frac{2}{3})^\tau$

Answer: $\underline{\mathsf{E}(X_t') = 0 \text{ and } r_{X'}(\tau) = \frac{9}{20}(\ln \frac{2}{3})^2 (\frac{2}{3})^\tau}$

□

6. Let the expectation function of $\{X_t\}$ be $m_X = 0$ and the covariance function be $R_X(\tau) = \max(0, \frac{1}{2} - |\tau|)$.

(a) Calculate the spectral density $S_X(f)$. (By all means, use the table!) (3p)

(b) The signal $\{Y_t\}$ is achieved by a linear transfer defined by $Y_t = \int_{t-1}^{t} X_u\,du$. Derive the spectral density of $\{Y_t\}$. (5p)

Solution:

(a) $m_X = 0$ and $R_X(\tau) = \max(0, \frac{1}{2} - |\tau|) = \begin{cases} \frac{1}{2} - |\tau| & \text{om } -\frac{1}{2} < \tau < \frac{1}{2} \\ 0 & \text{annars} \end{cases}$

According to the table we have that

$$g(\tau) = \begin{cases} 1 - \alpha|\tau| & \text{if } |\tau| \le \frac{1}{\alpha} \\ 0 & \text{otherwise} \end{cases} \Rightarrow G(f) = \begin{cases} \frac{1}{\alpha} & \text{if } f = 0 \\ \frac{2\alpha}{(2\pi f)^2}(1 - \cos(\frac{2\pi f}{\alpha})) & \text{if } f \neq 0 \end{cases}$$

Now let $g(\tau) = \max(0, 1 - 2|\tau|)$ because then

$$G(f) = \begin{cases} \frac{1}{2} & \text{om } f = 0 \\ \frac{1-\cos(\pi f)}{(\pi f)^2} & \text{om } f \neq 0 \end{cases}$$

Furthermore $\frac{1}{2}g(\tau) = \frac{1}{2}\max(0, 1 - 2|\tau|) = \max(0, \frac{1}{2} - |\tau|) = r_X(\tau)$ so since $G(f) = \int_{\mathbb{R}} e^{-i2\pi f\tau} g(\tau)\,d\tau$ the spectral density is

$S_X(f) = \int_{\mathbb{R}} e^{-i2\pi f\tau}\frac{1}{2}g(\tau)\,d\tau = \frac{1}{2}\int_{\mathbb{R}} e^{-i2\pi f\tau} g(\tau)\,d\tau = \frac{1}{2}G(f) =$

$$= \begin{cases} \frac{1}{4} & \text{if } f = 0 \\ \frac{1-\cos(\pi f)}{2(\pi f)^2} & \text{if } f \neq 0 \end{cases}$$

(b) In order to derive the spectral density, S_Y, of the transferred signal we may derive the impulse response, h, and further the corresponding frequency function, H, and we will have the desired spectral density S_Y by the relation $S_Y = |H|^2 S_X$.

The impulse response is recognised by inspection of the linear transfer: $Y_t = \int_{-\infty}^{\infty} h(u)X_{t-u}\,du$ so for the process $\{Y_t\}$ in this case we have that

$$Y_t = \int_{t-1}^{t} X_u\,du = \left\{ \begin{array}{ll} v = t - u & u = t \Rightarrow v = 0 \\ dv = -du & u = t - 1 \Rightarrow v = 1 \end{array} \right\} = \int_1^0 X_{t-v}\,(-dv) =$$

$$= \int_0^1 X_{t-v}\,dv = \int_{-\infty}^{\infty} h(v)X_{t-v}\,dv \text{ if } h(v) = \begin{cases} 1 & \text{if } 0 < v < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore the frequency function is

$H(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} h(\tau)\,d\tau = \int_0^1 e^{-i2\pi f\tau}\cdot 1\,d\tau = [\frac{e^{-i2\pi f\tau}}{-i2\pi f}]_0^1 = \frac{1-e^{-i2\pi f}}{i2\pi f}$ if $f \neq 0$.

If $f = 0$ then $H(f) = \int_0^1 e^0 \cdot 1\,d\tau = 1$ i.e. $H(f) = \begin{cases} \frac{1-e^{-i2\pi f}}{i2\pi f} & \text{if } f \neq 0 \\ 1 & \text{if } f = 0 \end{cases}$

$$S_Y(f) = |H(f)|^2 S_X(f) = \begin{cases} \left|\frac{1-e^{-i2\pi f}}{i2\pi f}\right|^2 \frac{1-\cos(\pi f)}{2(\pi f)^2} & \text{if } f \neq 0 \\ |1|^2 \cdot \frac{1}{4} & \text{if } f = 0 \end{cases}$$

Assume $f \neq 0$. Then with $e^{i\alpha} = \cos\alpha + i\sin\alpha$ we have that $S_Y(f) =$

$$= \left|\frac{1-\cos(2\pi f)+i\sin(2\pi f)}{i2\pi f}\right|^2 \frac{1-\cos(\pi f)}{2(\pi f)^2} = \frac{(1-\cos(2\pi f))^2+\sin^2(2\pi f)}{(2\pi f)^2}\cdot\frac{1-\cos(\pi f)}{2(\pi f)^2} = \frac{(1-\cos(2\pi f))(1-\cos(\pi f))}{4(\pi f)^4}$$

Answer: $S_Y(f) = \begin{cases} \dfrac{(1-\cos(2\pi f))(1-\cos(\pi f))}{4(\pi f)^4} & \text{om } f \neq 0 \\ 1/4 & \text{om } f = 0 \end{cases}$

□