Solutions to the Exam for Random Processes, 5p

August 13, 2003, 9.00 am − 13.00 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

For each problem a *complete* solution should be given.
All solutions should be thoroughly presented.
Each soloution should start at the top of a new sheet of paper.
Only one solution a sheet.

1. Show that if $\boldsymbol{X}$ is n-dimensionally normally distributed with uncorrelated vector elements, then the vector elements are independent. (3p)

   Solution:
   (See Lindgren Rotzén p 87−88.) □

2. Show that if $\{X_t\}$ is weakly stationary, H is a transfer function for a linear filter, $\{Y_t\}$ is the filtered process and S_X and S_Y are the respective spectral densities, then $S_Y(f) = |H(f)|^2 S_X(f)$. (4p)

   Solution:
   (See Lindgren Rotzén p 109.) □

3. Let $\{X_t\}$ be a Poisson process with intensity $\lambda = 0.02$ and calculate
   (a) $P(X_t - X_{t-25} > 1)$ for $t \geq 26$. (2p)
   (b) $P(X_{1000}^2 \leq 100)$ approximately. (2p)
   (c) Let $Y_t = (X_t - 0.02t)/\sqrt{0.02t}$ for $t \geq 1$. Is $\{Y_t\}$ a stationary process? If so, show that. If not, give a counterexample. (3p)

   Solution:
   (a) $X_t \in Poi(0.02t)$. Let $X_t = \sum_{s=1}^{t} Z_s$ where $Z_1, Z_2, \ldots, Z_t$ are independent variables each distributed $Z_s \in Poi(0.02)$. Then $P(X_t - X_{t-25} > 1) =$
   $= P(\underbrace{\sum_{s=t-24}^{t} Z_t}_{\in Poi(0.02 \cdot 25)} > 1) = 1 - P(\sum_{s=t-24}^{t} Z_t \leq 1) = 1 - 0.910 = \underline{0.09}$

   (b) $E(X_{1000}) = V(X_{1000}) = 1000 \cdot 0.02 = 20$. According to the Central Limit Theorem, approximately $X_{1000} = \sum_{t=1}^{1000} Z_t \in N(20, \sqrt{20})$. Thus $P(X_{1000}^2 \leq 100) =$
   $= P(X_{1000} \leq 10) = \Phi(\frac{10-20}{\sqrt{20}}) = \Phi(-2.2361) = 1 - \Phi(2.24) = \underline{0.0124}$.

(c) The mean $E(Y_t) = 0$ and $V(Y_t) = 1$ independent of t which does not violate the conditions of weak stationarity. To see that $\{Y_t\}$ really is *not* weakly stationary we have to look at the covariance function. We find that $C(Y_t, Y_{t+1}) =$
$= C\Big((X_t - 0.02t)/\sqrt{0.02t}\ ,\ (X_{t+1} - 0.02(t+1))/\sqrt{0.02(t+1)}\Big) =$
$= \frac{1}{\sqrt{0.02t}\sqrt{0.02(t+1)}} C(X_t, X_{t+1}) = \frac{1}{0.02\sqrt{t(t+1)}} C(\sum_{s=1}^{t} Z_s\ ,\ \sum_{s=1}^{t+1} Z_s) =$
$= \frac{1}{0.02\sqrt{t(t+1)}} C(\sum_{s=1}^{t} Z_s\ ,\ \sum_{s=1}^{t} Z_s + Z_{t+1}) = \frac{1}{0.02\sqrt{t(t+1)}} (V(X_t) + \underbrace{C(X_t, Z_{t+1})}_{=0}) =$
$= \frac{1}{0.02\sqrt{t(t+1)}} \cdot 0.02t = \sqrt{\frac{t}{t+1}}$ which is a function of t. Therefore $\{Y_t\}$ is not weakly stationary and since a random process cannot be strictly stationary unless it is weakly stationary, $\{Y_t\}$ is not stationary in any sense.

□

4. Let $\{X_t\}$ be an $MA(2)$-process defined by $X_t = \epsilon_t + 0.5\epsilon_{t-1} - 2\epsilon_{t-2}$ where $\{\epsilon_t\}$ is white noise with $\sigma_\epsilon^2 = 1$. Calculate
   (a) $P(X_t - X_{t-1} > 1)$. (3p)
   (b) the covariance function $R_X(\tau)$. (3p)

Solution:

(a) $P(X_t - X_{t-1} > 1) = P((\epsilon_t + 0.5\epsilon_{t-1} - 2\epsilon_{t-2}) - (\epsilon_{t-1} + 0.5\epsilon_{t-2} - 2\epsilon_{t-3}) > 1) =$
$= P(\epsilon_t - 0.5\epsilon_{t-1} - 2.5\epsilon_{t-2} + 2\epsilon_{t-3} > 1)$. Since $E(\epsilon_t) = 0$ for all t we have that $E(\epsilon_t - 0.5\epsilon_{t-1} - 2.5\epsilon_{t-2} + 2\epsilon_{t-3}) = 0$. Since $V(\epsilon_t) = 1$ we also have that $V(\epsilon_t - 0.5\epsilon_{t-1} - 2.5\epsilon_{t-2} + 2\epsilon_{t-3}) = V(\epsilon_t) + 0.5^2 V(\epsilon_{t-1}) + 2.5^2 V(\epsilon_{t-2}) + 2^2 V(\epsilon_{t-3}) =$
$= 1 + 0.25 + 6.25 + 4 = 11.5$. Thus $\epsilon_t - 0.5\epsilon_{t-1} - 2.5\epsilon_{t-2} + 2\epsilon_{t-3} \in N(0, \sqrt{11.5})$ and $P(X_t - X_{t-1} > 1) = 1 - P(\epsilon_t - 0.5\epsilon_{t-1} - 2.5\epsilon_{t-2} + 2\epsilon_{t-3} \leq 1) = 1 - \Phi(\frac{1-0}{\sqrt{11.5}}) =$
$= 1 - \Phi(0.29) = \underline{0.3859}$

(b) $R_X(0) = V(\epsilon_t + 0.5\epsilon_{t-1} - 2\epsilon_{t-2}) = 5.25$,
$R_X(1) = C(\epsilon_t + 0.5\epsilon_{t-1} - 2\epsilon_{t-2}, \epsilon_{t-1} + 0.5\epsilon_{t-2} - 2\epsilon_{t-3}) = 0.5V(\epsilon_{t-1} - 2V(\epsilon_{t-2}) = 1$,
$R_X(2) = C(\epsilon_t + 0.5\epsilon_{t-1} - 2\epsilon_{t-2}, \epsilon_{t-2} + 0.5\epsilon_{t-3} - 2\epsilon_{t-4}) = -2V(\epsilon_{t-2}) = -2$
and $R_X(\tau) = 0$ for $|\tau| \geq 3$.
Thus $\underline{R_X(\tau) = 5.25\delta_0(\tau) + \delta_{-1}(\tau) + \delta_1(\tau) - 2(\delta_{-2}(\tau) + \delta_2(\tau))}$.

□

5. A signal has spectral density $f^2 e^{-c|f|}$. The signal is to be sampled from with sampling interval 0.5. Determine approximately the value of c such that the error margin of the sampled signal is less than 5%. (4p)

Solution:

$$0.05 = \frac{4 \text{ tailparts}}{\text{whole signal}} = \frac{4\int_{1/(2\cdot 0.5)}^{\infty} S_X(f)\,df}{\int_{-\infty}^{\infty} S_X(f)\,df}$$

where $4\int_1^\infty S_X(f)\,df = 4\int_1^\infty f^2 e^{-cf}\,df = 4([f^2(-\frac{1}{c})e^{-cf}]_1^\infty - \int_1^\infty 2f\cdot(-\frac{1}{c})e^{-cf}\,df) =$
$= 4(\frac{1}{c}e^{-c}+\frac{2}{c}([f\cdot(-\frac{1}{c})e^{-cf}]_1^\infty - \int_1^\infty 1\cdot(-\frac{1}{c})e^{-cf}\,df)) = 4(\frac{1}{c}e^{-c}+\frac{2}{c^2}e^{-c}+\frac{2}{c^2}[(-\frac{1}{c})e^{-cf}]_1^\infty) =$
$= \frac{4e^{-c}}{c^3}(c^2+2c+1)$
and $\int_{-\infty}^\infty S_X(f)\,df = 2\int_0^\infty f^2 e^{-cf}\,df =$
$= 2([f^2\cdot(-\frac{1}{c})e^{-cf}]_0^\infty + \frac{2}{c}([f\cdot(-\frac{1}{c})e^{-cf}]_0^\infty + \frac{1}{c}[1\cdot(-\frac{1}{c})e^{-cf}]_0^\infty)) = 2(0+\frac{2}{c}(0+\frac{1}{c}\cdot\frac{1}{c})) = \frac{4}{c^3}$
so $0.05 = \frac{\frac{4e^{-c}}{c^3}(c^2+2c+1)}{\frac{4}{c^3}} = e^{-c}(c^2+2c+1)$ which is a non-linear equation. Evidently $f(c) = e^{-c}(c^2+2c+1) \searrow 0$ as $c\to\infty$ and since $f(0)=1$ and f is continuous there must be (at least one) value of c such that $f(c) = 0.05$. Experimenting with a few values give that

c	$f(c)$
1	1.47
9	0.01234
7	0.05836
7.5	0.03996
7.2	0.05020
7.3	0.04654

so with $c \geq 7.3$ the error margin is garateed to be less than 5%.

6. Let $\{X_t\}$ be a process with mean $m_X = 0$ and spectral density $S_X(f) = \frac{1}{2}\delta_{-1}(f) + \frac{1}{2}\delta_1(f)$. Calculate the covariance function of
   (a) $\{X_t\}$. (3p)
   (b) the derivative process $\{X_t'\}$. (3p)

Solution:

(a) $R_X(\tau) = \int_{\mathbb{R}} e^{i2\pi f\tau} S_X(f)\,df = \int_{\mathbb{R}} e^{i2\pi f\tau}(\frac{1}{2}\delta_{-1}(f)+\frac{1}{2}\delta_1(f))\,df =$
$= \{\text{Eulers formler}\} = \frac{1}{2}(e^{i2\pi(-1)\tau} + e^{i2\pi\cdot 1\cdot\tau}) = \underline{\cos(2\pi\tau)}$

(b) Since the process $\{X_t\}$ is weakly stationary the covariance function of the dervative process may be calculated as $-R_X''(\tau)$. $R_X'(\tau) = \frac{1}{2}\cdot i2\pi(-e^{-i2\pi\tau}+e^{i2\pi\tau}) =$
$= 2\pi\sin(2\pi\tau)$ and $R_X''(\tau) = i2\pi\cdot(i\pi)(e^{-i2\pi\tau}+e^{i2\pi\tau}) = -4\pi^2\cos(2\pi\tau)$.
Thus the covariance function of the derivative process is $\underline{-R_X''(\tau) = 4\pi^2\cos(2\pi\tau)}$.

□