Solutions to the Exam for Random Processes, 5p

October 25, 2002, 9.00 am − 13.00 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids:
Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that all weakly stationary stationary normal processes are strictly stationary. (4p)

   Lösning:
   (See Lindgren-Rotzén, p 89.)

2. Assume $\{X_t\}$ is a weakly stationary process differentiable in squared mean and that $\{X_t\}$ has a $R_X(\tau)$. Show that $M_{X'}(t) = 0$ och att $R_{X'}(\tau) = -R''_X(\tau)$. (4p)

   Solution:
   (Se Lindgren-Rotzén, p 127)

3. Let $\{X_t\}$ be a Poisson process with intensity 0.1 and let $\{Y_t\}$ be defined by $Y_t = 4X_{t-1} - 3X_t - 7,\ t = 2, 3, 4, \ldots$
   (a) Calculate the covariance functionen of $\{Y_t\}$. (3p)
   (b) Calculate $P(3Y_{151} + 4Y_{150} > 16X_{149})$ approximately. (4p)

Solution:

(a) $\{X_t\}$ Poisson process $\Rightarrow$ $R_X(s,t) = 0.1 \min(s,t)$.

$$
\begin{aligned}
R_Y(s,t) &= C(4X_{s-1} - 3X_s + 7\ ,\ 4X_{t-1} - 3X_t + 7) \\
&= 16C(X_{s-1}, X_{t-1}) - 12C(X_s, X_{t-1}) - 12C(X_{s-1}, X_t) + 9C(X_s, X_t) \\
&= 1.6 \min(s-1, t-1) - 1.2 \min(s, t-1) - 1.2 \min(s-1, t) + 0.9 \min(s,t) \\
&= 2.5 \min(s,t) - 1.6 - 1.2 \min(s, t-1) - 1.2 \min(s-1, t) \\
&= \begin{cases} 2.5s - 1.6 - 1.2s - 1.2(s-1) & \text{if } s \le t-1 \\ 2.5s - 1.6 - 1.2(t-1) - 1.2(s-1) & \text{if } s = t \\ 2.5t - 1.6 - 1.2(t-1) - 1.2t & \text{if } s \ge t+1 \end{cases} \\
&= \begin{cases} 0.1s - 0.4 & \text{if } s \le t-1 \\ 0.1s + 0.8 & \text{if } s = t \\ 0.1t - 0.4 & \text{if } s \ge t+1 \end{cases} \\
&= 0.1 \min(s,t) + 1.2\delta_0(s-t) - 0.4
\end{aligned}
$$

(b) Since $\{X_t\} = \{X_1, X_2, X_3, \ldots\}$ is a Poisson process it may be represented $\{Z_1, Z_1 + Z_2, Z_1 + Z_2 + Z_3, \ldots\}$ where $\{Z_t\}$ is a sequence of independent variables and each $Z_t \in Poi(0.1)$. Therefore

$$
\begin{aligned}
P(3Y_{151} + 4Y_{150} > 16X_{149}) &= P(3(4X_{150} - 3X_{151} + 7) + 4(4X_{149} - 3X_{150} + 7) > 16X_{149}) \\
&= P(12X_{150} - 9X_{151} + 21 + 16X_{149} - 12X_{150} + 28 > 16X_{149}) \\
&= P(-9X_{151} + 49 > 0) \\
&= P(X_{151} < 49/9)
\end{aligned}
$$

Since $X_{151} = \sum_{k=1}^{151} Z_k$ we have approximately that $X_{151} \in N(151 \cdot 0.1, \sqrt{151 \cdot 0.1}) =$ $= N(15.1, 3.8859)$ according to CLT, and hence

$$
\begin{aligned}
P(3Y_{151} + 4Y_{150} > 16X_{149}) &= \Phi\Big(\frac{49/9 - 15.1}{3.8859}\Big) \\
&= 1 - \Phi(2.48) \\
&= 0.0066
\end{aligned}
$$

4. Let $\{X_t\}$ be an AR(2)-process defined by $X_t = aX_{t-1} - 0.5X_{t-2} + \epsilon_t$.
   (a) What is the condition on a for $\{X_t\}$ to be stationary? (4p)
   (b) Calculate a if $V(X_t) = 7$ and $C(X_t, X_{t+1}) = -5.6$. (2p)

Solution:

(a) Taking the variance of both sides gives us

$$\begin{aligned} \sigma_X^2 &= V(X_t) \\ &= V(aX_{t-1} - 0.5X_{t-2} + \epsilon_t) \\ &= a^2V(X_{t-1}) - aC(X_{t-1}, X_{t-2}) + 0.25V(X_{t-2}) + \sigma_\epsilon^2 \\ &= (0.25 + a^2)\sigma_X^2 - ar(1) + \sigma_\epsilon^2 \quad (*) \end{aligned}$$

$$\begin{aligned} r(1) &= C(X_t, X_{t-1}) \\ &= C(aX_{t-1} - 0.5X_{t-2} + \epsilon_t, X_{t-1}) \\ &= aV(X_{t-1}) - 0.5r(1) \end{aligned}$$

$$\Rightarrow r(1)(1 + 0.5) = a\sigma_X^2$$
$$\Rightarrow r(1) = \tfrac{a}{1.5}\sigma_X^2$$

Inserting this into $(*)$ renders
$\sigma_X^2 = (0.25 - a^2)\sigma_X^2 - a \cdot \frac{a}{1.5}\sigma_X^2 + \sigma_\epsilon^2$
i.e. $\sigma_X^2(1 - (0.25 + a^2 - \frac{a^2}{1.5})) = \sigma_\epsilon^2$.
But since the variance of X_t is always positive we have that

$$0 < \sigma_X^2 = \frac{\sigma_\epsilon^2}{0.75 - a^2 + \frac{2}{3}a^2} = \frac{12\sigma_\epsilon^2}{9 - 4a^2}.$$

Here the right side is also $\sigma_\epsilon^2 > 0$ and hence the denominator $9 - 4a^2$ must be positive for the quotient to be positive! This means that

$$9 - 4a^2 > 0 \quad \Rightarrow \quad 9 > 4a^2 \quad \Rightarrow \quad |a| < \tfrac{3}{2}$$

i.e. $\underline{-\frac{3}{2} < a < \frac{3}{2}}$.

(b) From the calculations above we have that

$$r(1) = \tfrac{a}{1.5}\sigma_X^2 \quad \Rightarrow \quad a = \tfrac{r(1)\cdot 1.5}{\sigma_X^2} = \tfrac{-5.6\cdot 1.5}{7} = \underline{-1.2}.$$

Alternatively if not having made the calculations in part (a), we have according to the Yule-Walker equations that $r_X(\tau) + a_1 r_X(\tau-1) + a_2 r_X(\tau-2) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = 1, 2, \ldots \end{cases}$
With $a_1 = -a$ and $a_2 = 0.5$ we have for $\tau = 1$ that $r_X(-1) - ar_X(0) + 0.5r_X(1) = 0$. The variance is $r(0) = 7$ and the covariance is $r(1) = -5.6$ and thus

$$-5.6 - 7a - 5.6 = 0 \quad \text{i.e. } a = \frac{-5.6 - 0.5 \cdot 5.6}{7} = \underline{-1.2}.$$

5. A signal has spectral density $S(f) = 1/(1+4f^2)$. How large proportion of the signal is transferred correctly for certain if it is sampled with interval 0.1? (4p)

Solution:
Approximately 4 "tailparts " are distorted and
[whole signal] − [4 tailparts]
is transferred correctly, so the proportion is

$$\frac{[\text{whole signal}] - [\text{4 tailparts}]}{[\text{whole signal}]} = 1 - \frac{4\int_{1/2\cdot 0.1}^{\infty} R(f)\,df}{\int_{-\infty}^{\infty} R(f)\,df}$$

where

$$\begin{aligned}\int_{1/2\cdot 0.1}^{\infty} R(f)\,df &= \tfrac{1}{2}\int_{1/0.2}^{\infty} \tfrac{2}{1+4f^2}\,df \\ &= \tfrac{1}{2}[\arctan(2f)]_5^{\infty} \\ &= \tfrac{1}{2}(\tfrac{\pi}{2} - \arctan(10))\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} R(f)\,df &= \int_0^{\infty} \tfrac{2}{1+4f^2}\,df \\ &= [\arctan(2f)]_0^{\infty} \\ &= \tfrac{\pi}{2}\end{aligned}$$

Thus the proportion is

$$1 - \frac{4\cdot\frac{1}{2}(\frac{\pi}{2} - \arctan(10))}{\frac{\pi}{2}} = 1 - \tfrac{2}{\pi}(\pi - 2\arctan(10)) = \tfrac{4}{\pi}\arctan(10) - 1 = \underline{0.873}$$

6. Let $\{X_t\}$ be a weakly stationary process with expectation function $M_X = \frac{1}{2}$ and covariance function $R_X(\tau) = \begin{cases} \frac{1}{2}|\tau| & \text{if } |\tau| \leq 2 \\ 0 & \text{otherwise.} \end{cases}$ Assume further that $\{X_t\}$ is transferred with impulse response $h(t) = \delta_{-1}(t) + \delta_1(t)$.

(a) Is the transfer causal? (1p)

(b) Calculate expectation function and covariance function of the transferred signal. (4p)

Lösning:

(a) No, the transfer is *not* causal since $h(-1) = 1$ i.e. $h(t) \not\equiv 0$ for all $t < 0$.

(b)
$$\begin{aligned} M_Y(t) &= E(\textstyle\int_{\mathbb{R}} h(u)X_{t-u}du \\ &= \textstyle\int_{\mathbb{R}}(\delta_{-1}(u) + \delta_1(u))m_X\, du \\ &= M_X(1+1) \\ &= \tfrac{1}{2}\cdot 2 \;=\; 1 \end{aligned}$$

$$\begin{aligned} R_Y(\tau) &= \textstyle\int\int_{\mathbb{R}^2} h(u)h(v)R_X(\tau+u-v)\, dudv \\ &= \textstyle\int\int_{\mathbb{R}^2}(\delta_{-1}(u)+\delta_1(u))(\delta_{-1}(v)+\delta_1(v))R_X(\tau+u-v)\, dudv \\ &= \textstyle\int\int_{\mathbb{R}^2}(\delta_{-1}(u)\delta_{-1}(v)+\delta_{-1}(u)\delta_1(v)+\delta_1(u)\delta_{-1}(v)+\delta_1(u)\delta_1(v))R_X(\tau+u-v)\, dudv \\ &= R_X(\tau-1-(-1)) + R_X(\tau-1-1) + R_X(\tau+1-(-1)) + R_X(\tau+1-1) \\ &= 2R_X(\tau) + R_X(\tau-2) + R_X(\tau+2) \end{aligned}$$

[Correct answer so far in the (b) part renders 2 points.]

We need to examine the intervals

$$\begin{aligned} \text{I}: &\quad |\tau+2| \leq 2 \;\Leftrightarrow\; -2 \leq \tau+2 \leq 2 \;\Leftrightarrow\; -4 \leq \tau \leq 0 \\ \text{II}: &\quad |\tau| \leq 2 \;\Leftrightarrow\; -2 \leq \tau \leq 2 \;\Leftrightarrow\; -2 \leq \tau \leq 2 \\ \text{III}: &\quad |\tau-2| \leq 2 \;\Leftrightarrow\; -2 \leq \tau-2 \leq 2 \;\Leftrightarrow\; 0 \leq \tau-2 \leq 4 \end{aligned}$$

which renders the subintervals

$\text{I}: [-4,-2),\ \text{I}\wedge\text{II}: [-2,0),\ \text{I}\wedge\text{II}\wedge\text{III}: \{0\},\ \text{II}\wedge\text{III}: (0,2],\ \text{III}: (2,4]$

$$r_Y(\tau) = 2r_X(\tau) + r_X(\tau-2) + r_X(\tau+2)$$

$$= \begin{cases} \frac{1}{2}|\tau+2| & \text{if } -4 \leq \tau < -2 \\ |\tau| + \frac{1}{2}|\tau+2| & \text{if } -2 \leq \tau < 0 \\ |\tau| + \frac{1}{2}(|\tau+2| + |\tau-2|) & \text{if } \tau = 0 \\ |\tau| + \frac{1}{2}|\tau-2| & \text{if } 0 < \tau \leq 2 \\ |\tau-2| & \text{if } 2 < \tau \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$= \begin{cases} -\frac{1}{2}\tau - 1 & \text{if } -4 \leq \tau < -2 \\ -\frac{1}{2}\tau + 1 & \text{if } -2 \leq \tau < 0 \\ 2 & \text{if } \tau = 0 \\ \frac{1}{2}\tau + 1 & \text{if } 0 < \tau \leq 2 \\ \frac{1}{2}\tau - 1 & \text{if } 2 < \tau \leq 4 \\ 0 & \text{otherwise} \end{cases}$$