Solutions to the Exam for Random Processes, 7.5 ECTS

December 19, 2003, 9.00 am − 1.00 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that <u>if</u> $\{X_t\}$ is an $AR(p)$ process, <u>then</u> its spectral density function is

$$R_X(f) = \frac{\sigma_\epsilon^2}{|\sum_{k=0}^{p} a_k e^{-i2\pi fk}|^2} \qquad \text{(3p)}$$

Solution: (See Theorem 12 in the book.) □

2. Show that <u>if</u> $\{X_t\}$ is weakly stationary and differentiable, <u>then</u> $E(X_t') = 0$. (3p)

Solution: (See Theorem 17 in the book.) □

3. Let $X = \begin{cases} 0 & \text{w.p. } 1/3 \\ 1 & \text{w.p. } 2/3 \end{cases}$ and independently $\{Y_t\}$ be a Poisson process with intensity 0.05. Then what is

   (a) $P(X > Y_3)$? (2p)

   (b) $E((-1)^{X+Y_{40}})$? (Hint: $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda$ and $\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \frac{1}{2}(e^\lambda + e^{-\lambda})$) (3p)

Solution:

(a) $P(X > Y_3) = P(X = 1, Y_3 = 0) = \frac{2}{3} \cdot e^{-3\cdot 0.05} \frac{(0.05\cdot 3)^0}{0!} = \frac{2}{3} e^{-0.15} = 0.574$

(b) Since $X \perp Y_{40}$ we have that

$$E((-1)^{X+Y_{40}}) = E((-1)^X)E((-1)^{Y_{40}})$$

where

$$E((-1)^X) = \tfrac{1}{3}(-1)^0 + \tfrac{2}{3}(-1)^1 = \tfrac{1}{3} - \tfrac{2}{3} = -\tfrac{1}{3}$$

and since $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \Rightarrow \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} =$ $= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} - \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = e^\lambda - \frac{1}{2}(e^\lambda + e^{-\lambda}) = \frac{1}{2}(e^\lambda - e^{-\lambda})$ we have that

$$\begin{aligned} E((-1)^{Y_{40}}) &= \sum_{k=0}^{\infty} (-1)^k \frac{(40\cdot 0.05)^k}{k!} e^{-40\cdot 0.05} \\ &= e^{-2}\Big(\sum_{k=0}^{\infty} (-1)^{2k} \tfrac{2^{2k}}{(2k)!} + \sum_{k=0}^{\infty} (-1)^{2k+1} \tfrac{2^{2k+1}}{(2k+1)!}\Big) \\ &= e^{-2}\Big(\sum_{k=0}^{\infty} \tfrac{2^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \tfrac{2^{2k+1}}{(2k+1)!}\Big) \end{aligned}$$

$$= e^{-2}\Big(\tfrac{1}{2}(e^2+e^{-2}) - \tfrac{1}{2}(e^2-e^{-2})\Big)$$
$$= e^{-4}$$

Summing up, we have found that

$$E((-1)^{X+Y_{40}}) = E((-1)^X)E((-1)^{Y_{40}}) = -\frac{1}{3}\,e^{-4} = -0.0061 \qquad \square$$

4. Let $\{X_t\}$ be defined by the relation $X_t = \epsilon_t - \epsilon_{t-1} + 2\epsilon_{t-2}$ for all $t \in \mathbb{Z}$ where $\{\epsilon_t\}$ is a sequence of independent variables all distributed $N(1,1)$. Calculate $P(|\frac{X_t+X_{t-1}-X_{t-2}}{3}| < 2)$. (3p)

Solution: Let $Y_t = X_t + X_{t-1} - X_{t-2}$.
Then $Y_t = (\epsilon_t - \epsilon_{t-1} + 2\epsilon_{t-2}) + (\epsilon_{t-1} - \epsilon_{t-2} + 2\epsilon_{t-3}) - (\epsilon_{t-2} - \epsilon_{t-3} + 2\epsilon_{t-4}) =$
$= \epsilon_t + 3\epsilon_{t-3} - 2\epsilon_{t-4}$. Therefore $m_Y = E(\epsilon_t) + 3E(\epsilon_{t-3}) - 2E(X_{t-4}) = 2$
and since $\{\epsilon_t\}$ are independent, $C(\epsilon_s, \epsilon_t) = 0$ whenever $s \neq t$, which is why
$V(Y_t) = V(\epsilon_t) + 9V(\epsilon_{t-3}) + 4V(X_{t-4}) = 14$. Thus $Y_t \in N(2,14)$.
Hence we have that $P(|\frac{X_t+X_{t-1}-X_{t-2}}{3}| < 2) = P(-2 < \frac{Y_t}{3} < 2) =$
$= P(Y_t < 2{\cdot}3) - P(Y_t < -2{\cdot}3) = \Phi(\frac{6-2}{\sqrt{14}}) - \Phi(\frac{-6-2}{\sqrt{14}}) = \Phi(\frac{4}{\sqrt{14}}) + \Phi(\frac{8}{\sqrt{14}}) - 1 = 0.8415$ $\square$

5. A signal $\{X_t : t \in \mathbb{R}\}$ has cvf $r(\tau) = \frac{1}{1+\tau^2}$. The cost of sampling from $\{X_t\}$ is described by $C(d) = e^{\pi/d}(2\pi d + \frac{1}{d})$ where d is the lenght of the sampling interval. What is the minimal value of the product, $Err(d)C(d)$, of error proportion due to the alias effect, $Err(d)$, and sampling cost, $C(d)$? (4p)

Solution: $r(\tau) = \frac{1}{1+\tau^2} \;\Rightarrow\; R(f) = \pi e^{-2\pi|f|}$.

$$Err(d) = \frac{\text{4 tail-parts}}{\text{total effect}} = \frac{4\int_{1/2d}^{\infty} R(f)df}{\int_{\mathbb{R}} R(f)df}$$

where $\int_{\mathbb{R}} R(f)df = 2\pi \int_0^\infty e^{-2\pi f}df = 2\pi[\frac{e^{-2\pi f}}{-2\pi}]_0^\infty = (-0-(-e^0)) = 1$
so $Err(d) = 4\int_{1/2d}^\infty R(f)df = 4\pi[\frac{e^{-2\pi f}}{-2\pi}]_{1/2d}^\infty = \frac{4\pi}{2\pi}(-0-(-e^{-\pi/d})) = 2e^{-\pi/d}$.
Thus the product is $A(d) = Err(d)C(d) = 2e^{-\pi/d}e^{\pi/d}(2\pi d + \frac{1}{d}) = 4\pi d + \frac{2}{d}$. To find the minimum, differerentiate A w.r.t. d: $A'(d) = 4\pi - \frac{2}{d^2}$ and solve $A'(d) = 0$: $\frac{2}{d^2} = 4\pi \;\Rightarrow\; d = \sqrt{2/\pi}$ (since $d > 0$) and thus $d = \sqrt{2/\pi}$ is a stationary point of A. But is it a minimum? The second derivative w.r.t. d is $4/d^3$ which is positive for all $d > 0$ which means that $A(d)$ is convex in a neighbourhood of $\sqrt{2/\pi}$, and thus that $d = \sqrt{2/\pi}$ is indeed a minimum of A. Finally, we have that $\lim_{d\to 0} A(d) = \infty$ and $\lim_{d\to\infty} A(d) = \infty$, which is to say that $d = \sqrt{2/\pi}$ is the global minimum. The minimal product value is thus $A(\sqrt{2/\pi}) = 4\sqrt{2\pi} + \sqrt{2\pi} = 5\sqrt{2\pi} = 12.53$. $\square$

6. Let $\{W_t\}$ be a standard Wiener process (i.e. a Wiener process with $\sigma^2 = 1$) and let $X_t = e^{-t/2} W_{e^t}$ for all $t \in \mathbb{R}$. Then $\{X_t\}$ is a *Gauss-Markov process.*

   (a) Calculate expectation and covariance function of $\{X_t\}$. (3p)

   (b) Show that $\{X_t\}$ is strongly stationary. (3p)

   Let $Y_t = \int_{t-1}^{t+1} X_u \, du$ for all $t \in \mathbb{R}$. Calculate

   (c) the spectral density function of $\{Y_t\}$. (4p)

   (d) the probability $P(X_t < Y_t)$. (2p)

Solution:

(a) $\{W_t\}$ standard Wiener process $\Rightarrow$ $E(W_t) = 0, r_W(s,t) = \min(s,t)$.
$m_X(t) = e^{-t/2} E(W_{e^t})$.
$r_X(s,t) = C(e^{-s/2} W_{e^s}, e^{-t/2} W_{e^t}) = e^{-\frac{1}{2}(s+t)} r_W(e^s, e^t) = e^{-\frac{1}{2}(s+t)} \min(e^s, e^t) =$
$= \min(e^{\frac{1}{2}(s-t)}, e^{-\frac{1}{2}(s-t)}) = e^{\frac{1}{2}\min(s-t,-(s-t))} = e^{-\frac{1}{2}|s-t|}$

(b) $m_X = 0$ and $r_X(\tau) = e^{-\frac{1}{2}|\tau|}$. Thus $\{X_t\}$ is weakly stationary. Further, since $\{X_t\}$ is constructed by merely scaling and time translation of the Gaussian process $\{W_t\}$, the process $\{X_t\}$ is itself Gaussian. And since a weakly stationary Gaussian process is strongly stationary, $\{X_t\}$ is.

(c) We have that the spectral density $R_X(f) = \mathcal{F}(r_X) = \frac{4}{1+(4\pi f)^2}$ and the spectral density of the output may be derived from the input by using the relation $R_Y = |H|^2 R_X$ where H is the frequency function corresponding to the filtration changing X_t into Y_t. The impulse response rendering the filtration is derived by considering

$$Y_t = \int_{t-1}^{t+1} X_u \, du = \int_{\mathbb{R}} h(t-u) X_u \, du \quad \text{if} \quad h(t-u) = \begin{cases} 1 & \text{when } t-1 \le t-u \le t+1 \\ 0 & \text{o.w.} \end{cases}$$

But $t-1 \le t-u \le t+1 \;\Leftrightarrow\; -(t-1) \ge -(t-u) \ge -(t+1) \;\Leftrightarrow\; -1 \le u \le 1$,

i.e. $h(u) = \begin{cases} 1 & \text{when } -1 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$

Thus the frequency function is $H(f) = \mathcal{F}(h) = \int_{-1}^{1} e^{-i2\pi f t} dt = \frac{e^{-i2\pi f}}{-i2\pi f} - \frac{e^{i2\pi f}}{-i2\pi f} =$
$= \frac{1}{\pi f} \cdot \frac{e^{i2\pi f} - e^{-i2\pi f}}{2i} = \frac{1}{\pi f} \sin(2\pi f)$ if $f \neq 0$ and $H(0) = \lim_{f \to 0} H(f) = \ldots = 2$.
Thus the spectral density R_Y is
$R_Y(f) = |H(f)|^2 R_X(f) = \frac{1}{(\pi f)^2} \sin^2(2\pi f) \cdot \frac{4}{1+(4\pi f)^2} = \frac{4\sin^2(2\pi f)}{(\pi f)^2 + (2\pi f)^4}$.

(d) $P(X_t < Y_t) = P(X_t - Y_t \le 0)$.
Firstly we have that $E(X_t - Y_t) = E(X_t) - E(\int_{t-1}^{t+1} X_u \, du) = 0 - \int_{t-1}^{t+1} 0 \, du = 0$.
Secondly, $\{X_t\}$ is Gaussian and $\{Y_t\}$ is a linear transformation of $\{X_t\}$ so also Y_t and hence $X_t - Y_t$ are normally distributed.
Thus, due to the symmetry of the normal distribution about its mean,
$P(X_t - Y_t < 0) = P(X_t - Y_t > 0) = \frac{1}{2}$.

□