Solutions to Exam for Random Processes

December 18, 2004, 9 am – 1 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that if the process $\{X_t\}$ is differentiable, then the expectation function of the derivative process, $\{X_t'\}$, is $m_{X'} = 0$. (2p)

   Solution: (See the course literature, page 100.) □

2. Let $\{Y_t : t \in \mathbb{R}\}$ be the output process from filtering the input $\{X_t : t \in \mathbb{R}\}$ with impulse response h. Show that $V(Y_t) = \int_{\mathbb{R}} |H(f)|^2 R_X(f)\, df$ where H is the transfer function and R_X the spectral density of $\{X_t\}$. (3p)

   Solution: (See the course literature, page 84.) □

3. Is the process $\{Y_t : t = 2, 3, 4, \ldots\}$ weakly stationary if $Y_t = X_t + X_{t-1} - 2X_{t-2}$ for $t = 2, 3, 4, \ldots$ where $\{X_t\}$ is a Poisson process with parameter $\lambda = 1$? (4p)

   Solution: To be weakly stationary the ef, $m_Y(t)$, should be constant with respect to time and the cvf, $r_Y(s,t)$ should be a function of only the time distance $s-t$. $m_Y(t) = E(Y_t) = E(X_t + X_{t-1} - 2X_{t-2}) = E(X_t) + E(X_{t-1}) - 2E(X_{t-2}) = = t + (t-1) - 2(t-2) = 3$. Since $\{X_t\}$ is a Poisson process X_t may be written as a sum of independent Poisson distributed variables Z_k as $X_t = \sum_{k=1}^{t} Z_k$. Thus we have that

$$
\begin{aligned}
Y_t &= X_t + X_{t-1} - 2X_{t-2} \\
&= \sum_{k=1}^{t} Z_k + \sum_{k=1}^{t-1} Z_k - 2\sum_{k=1}^{t-2} Z_k \\
&= \Big(\sum_{k=1}^{t} Z_k - \sum_{k=1}^{t-2} Z_k\Big) + \Big(\sum_{k=1}^{t-1} Z_k - \sum_{k=1}^{t-2} Z_k\Big) \\
&= Z_t + Z_{t-1} + Z_{t-1} \\
&= Z_t + 2Z_{t-1}
\end{aligned}
$$

   and therefore the covariance function is

$$
\begin{aligned}
r_Y(s,t) &= C(Y_s, Y_t) \\
&= C(Z_s + 2Z_{s-1}, Z_t + 2Z_{t-1})
\end{aligned}
$$

$$= C(Z_s, Z_t) + 2C(Z_s, Z_{t-1}) + 2C(Z_{s-1}, Z_t) + 4C(Z_{s-1}, Z_{t-1})$$

$$= \begin{cases} 5 & \text{if } s = t \\ 2 & \text{if } s = t-1 \\ 2 & \text{if } s-1 = t \\ 0 & \text{o.w.} \end{cases}$$

This means that $r_Y(\tau) = \begin{cases} 5 & \text{if } \tau = 0 \\ 2 & \text{if } |\tau| = 1 \\ 0 & \text{o.w.} \end{cases}$

finally implying that $\{Y_t\}$ is weakly stationary. □

4. Let $\{X_t : t \in \mathbb{R}\}$ be a weakly stationary process with covariance function $r_X(\tau) = \frac{2}{1+4\pi^2\tau^2}$, and let $\{Y_t : t \in \mathbb{Z}\}$ be the process defined by sampling $\{X_t\}$ at integer timepoints. Calculate the spectral density function of $\{Y_t\}$. (4p)

Solution:

$$r_X(\tau) = \frac{2}{1+4\pi^2\tau^2} \quad \Rightarrow \quad R_X(f) = e^{-|f|} \quad \text{(according to tables)}$$

Since the sampling interval, d, is 1 we have that the spectral density of the sampled process is

$$\begin{aligned} R_Y(f) &= \sum_{k\in\mathbb{Z}} R_X\Big(f + \frac{k}{d}\Big) \\ &= \sum_{k\in\mathbb{Z}} e^{-|f+k|} \\ &= \underbrace{\sum_{k=-\infty}^{-1} e^{-|f+k|}}_{I} + e^{-|f|} + \underbrace{\sum_{k=1}^{\infty} e^{-|f+k|}}_{II} \end{aligned}$$

For the sampled process, Y_t, the interval where the spectral density is non-zero is $(-\frac{1}{2}, \frac{1}{2}]$ due to the alias effect.

$I:\ k \le -1$ and $f \le \frac{1}{2} \ \Rightarrow\ f + k < 0 \ \Rightarrow\ -|f+k| = f + k$ and hence

$$I = \sum_{k=-\infty}^{-1} e^{f+k} = e^f \sum_{k=-\infty}^{-1} e^k = e^f \sum_{k=1}^{\infty} e^{-k} = e^f\Big(\frac{1}{1-e^{-1}} - 1\Big) = \frac{e^{f-1}}{1-e^{-1}}$$

$II:\ k \ge 1$ and $f > -\frac{1}{2} \ \Rightarrow\ f + k > 0 \ \Rightarrow\ -|f+k| = -f - k$ and hence

$$II = \sum_{k=1}^{\infty} e^{-f-k} = e^{-f} \sum_{k=1}^{\infty} e^{-k} = e^{-f}\Big(\frac{1}{1-e^{-1}} - 1\Big) = \frac{e^{-f-1}}{1-e^{-1}}$$

Thus the spectral density is

$$R_Y(f) = \frac{e^{f-1} + e^{-f-1}}{1-e^{-1}} + e^{-|f|} = \frac{e^{-f} + e^f}{e-1} + e^{-|f|}, \quad f \in (-\tfrac{1}{2}, \tfrac{1}{2}]$$

□

5. Under the assumption that $Y_t = \sqrt{|W_t|}$ for all $t \in \mathbb{R}$ where $\{W_t\}$ is a Wiener process with parameter $\sigma^2 = 1$,

(a) calculate $P(Y_{100} \leq 3)$. (3p)

(b) derive the density function of Y_t. (4p)

Solution:

(a)

$$\begin{aligned} P(Y_{100} \leq 3) &= P(\sqrt{|W_{100}|} \leq 3) \\ &= P(|W_t| \leq 9) \\ &= P(-9 \leq W_t \leq 9) \\ &= P(W_t \leq 9) - P(W_t \leq -9) \\ &= \Phi(\tfrac{9}{\sqrt{100}}) - \Phi(-\tfrac{9}{\sqrt{100}}) \\ &= 2\Phi(\tfrac{9}{\sqrt{100}}) - 1 \\ &= 2\Phi(0.9) - 1 \\ &= 0.6318 \end{aligned}$$

(b)

$$\begin{aligned} F_{Y_t}(y) &= P(Y_t \leq y) \\ &= P(\sqrt{|W_t|} \leq y) \\ &= P(-y^2 \leq W_t \leq y^2) \\ &= P(-\tfrac{y^2}{\sqrt{t}} \leq \tfrac{W_t}{\sqrt{t}} \leq \tfrac{y^2}{\sqrt{t}}) \\ &= 2\int_0^{y^2/\sqrt{t}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2} dx \\ &= G(h(y)) \end{aligned}$$

where $G(h) = \int_0^h \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2} dx$ and $h(y) = y^2/\sqrt{t}$. Now, the density function of Y_t is $f_{Y_t}(y) = \frac{d}{dy} F_{Y_t}(y)$ which, according to the *chain rule*, is $\frac{d}{dy} G(h(y)) = G'(h(y))h'(y)$ and hence

$$f_{Y_t}(y) = 2\frac{1}{\sqrt{2\pi t}} \cdot e^{-\frac{1}{2t}(\frac{y^2}{\sqrt{t}})^2} \cdot \frac{2y}{\sqrt{t}} = 2\sqrt{\frac{2}{\pi}} \frac{y}{t} e^{-y^4/2t^2}, \; y > 0$$

□

6. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(1)$ process defined by $X_t + 0.7X_{t-1} = \epsilon_t$ for all $t \in \mathbb{Z}$ where $\epsilon_t \in N(0, \sigma_\epsilon^2)$ is white noise and $\epsilon_t \perp X_s$ for all $t > s$. Determine

(a) σ_ϵ^2 if $V(X_t) = 2$. (3p)

(b) the covariance function of $\{X_t\}$. (4p)

(c) $P(X_t + X_{t+1} \leq 1)$. (3p)

Solution:

(a) We have that

$$\begin{aligned} 2 &= V(X_t) \\ &= V(-0.7X_{t-1} + \epsilon_t) \\ &= (-0.7)^2 V(X_{t-1}) + V(\epsilon_t) \\ &= 0.49 \cdot 2 + \sigma_\epsilon^2 \end{aligned}$$

which implies that $\sigma_\epsilon^2 = 2(1 - 0.49) = 1.02$

(b) $r_X(0) = V(X_t) = 2$
$r_X(1) = C(X_t, X_{t-1}) = C(-0.7X_{t-1} + \epsilon_t, X_{t-1}) = -0.7V(X_{t-1}) = -0.7 \cdot 2$
$r_X(2) = C(-0.7X_{t-1} + \epsilon_t, X_{t-2}) = -0.7C(X_{t-1}, X_{t-2}) = (-0.7)^2 \cdot 2$
Continuing maybe a few steps more we get $r_X(3) = \ldots = (-0.7)^3 \cdot 2$, $r_X(4) = \ldots = (-0.7)^4 \cdot 2, \ldots$ and so we are led to guess that $r_X(\tau) = (-0.7)^\tau \cdot 2$ for $\tau \geq 0$.
Induction:
Initially for $\tau = 0$ we have that $r_X(0) = (-0.7)^0 \cdot 2$ (ok)
Under the assumption that $(*)$ $r_X(\tau) = (-0.7)^\tau \cdot 2$, we have to show that $r_X(\tau + 1) = (-0.7)^{\tau+1} \cdot 2$:

$$\begin{aligned} r_X(\tau + 1) &= C(X_t, X_{t-\tau-1}) \\ &= C(-0.7X_{t-1} + \epsilon_t, X_{t-\tau-1}) \\ &= -0.7r_X((t-1) - (t-\tau-1)) \\ &\overset{(*)}{=} -0.7(-0.7)^\tau \cdot 2 \\ &= (-0.7)^{\tau+1} \cdot 2 \end{aligned}$$

Due to the symmetry of the covariance function we finally have that

$$r_X(\tau) = 2(-0.7)^{|\tau|}$$

(c) Since $\{X_t\}$ is a Gaussian process, the sum $X_t + X_{t+1}$ is normally distributed. We have that $E(X_t + X_{t+1}) = E(X_t) + E(X_{t+1}) = 0$ and $V(X_t + X_{t+1}) = V(X_t) + V(X_{t+1}) + 2C(X_t, X_{t+1}) = 2 \cdot 2 + 2 \cdot (-0.7) \cdot 2 = 4(1 - 0.7) = 1.2$ and thus

$$P(X_t + X_{t+1} \leq 1) = \Phi\Big(\frac{1-0}{\sqrt{1.2}}\Big) = \Phi(0.913) = 0.8186$$

□