Solutions to Exam for Random Processes, 5 points/7.5 ECTS

August 14, 2004, 9 am – 1 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
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1. Show that if $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is n-dimensionally normally distributed with uncorrelated vector elements X_i, then $X_1, X_2, \ldots, X_n$ are all independent of each other. (3p)

   Solution: (See the book, p 64, Theorem 8, Section 4.) □

2. Assume that U is uniformly distributed on $[0, 1]$. Show that $X = \sqrt{-2\ln(1-U)}$ has density function $f(x) = x\,e^{-x^2/2}$ for all $x > 0$ (i.e. X is *Rayleigh distributed*). (3p)

   Solution: $P(\sqrt{-2\ln(1-U)} \leq x) \stackrel{x\geq 0}{=} P(-2\ln(1-U) \leq x^2) = P(\ln(1-U) \geq -\frac{x^2}{2}) =$
   $P(1-U \geq e^{-x^2/2}) = P(U \leq 1-e^{-x^2/2}) = \begin{cases} 0 & 1-e^{-x^2/2} < 0 \\ 1-e^{-x^2/2} & 0 \leq 1-e^{-x^2/2} \leq 1 \\ 1 & 1-e^{-x^2/2} > 1 \end{cases}$
   But $x > 0 \;\Rightarrow\; 0 < e^{-x^2/2} < 1 \;\Rightarrow\; 1-e^{-x^2/2} > 0$ and $1-e^{-x^2/2} < 1$. Thus $F_X(x) =$ $= P(\sqrt{-2\ln(1-U)} \leq x) = 1-e^{-x^2/2}$. Hence the density function is $f_X(x) = \frac{d}{dx}F_X(x) =$ $= \frac{d}{dx}(1-e^{-x^2/2}) = -(-\frac{2x}{2})e^{-x^2/2} = xe^{-x^2/2}$ which is the density of a Rayleigh distributed random variable. □

3. Let $\{X_t\}$ be an $AR(1)$ process defined by $X_t + 0.5X_{t-1} = \epsilon_t$ where $\{\epsilon_t\}$ is white noise and $V(X_t) = 2$. Determine the probability $P(X_t - 2X_{t-1} \leq 1)$. (3p)

   Solution: $P(X_t - 2X_{t-1} \leq 1) = ?$
   $E(X_t - 2X_{t-1}) = 0$ and $V(X_t - 2x_{t-1}) = C(X_t - 2X_{t-1}, X_t - 2X_{t-1}) =$
   $= C(X_t, X_t) + 4C(X_{t-1}, X_{t-1}) - 4C(X_t, X_{t-1}) = 2 + 8 - 4C(-0.5X_{t-1} + \epsilon_t, X_{t-1}) =$
   $= 10 - 4C(-0.5X_{t-1}, X_{t-1}) - 4C(\epsilon_t, X_{t-1}) = 14$. Since the AR-process $\{X_t\}$ is a Gaussian process this means that $X_t - 2X_{t-1} \in N(0, 14)$. Thus $P(X_t - 2X_{t-1} \leq 1) = \Phi(\frac{1-0}{\sqrt{14}}) =$
   $= \Phi(0.267) = 0.6064$. □

4. Let $\{X_t : t \in \mathbb{R}\}$ be a process with expectation function $m_X(t) = 1$ and covariance function $r_X(\tau) = e^{-|\tau|}$.

   (a) Calculate the spectral density function of $\{X_t\}$. (3p)

   $\{X_t\}$ is sampled: $Z_t = X_t$ for $t \in \mathbb{Z}$.

   (b) Calculate approximately the spectral density of the sampled process $\{Z_t\}$. (3p)

   Now, let $\{Y_t\}$ be defined by $Y_t = X_t + X_{t-1}$ for all $t \in \mathbb{R}$.

   (c) Show that $\{Y_t\}$ is weakly stationary. (4p)

   In an experiment, 8 observations of $\{X_t\}$ are about to be made.

   (d) Calculate approximately the variance of the mean value estimator. (3p)

   Suppose the values 1.73 1.85 2.25 1.89 0.82 1.99 1.71 2.11 are observed.

   (e) Estimate the expected value function $m_X(t)$. (2p)

Solution:

(a) $r_X(\tau) = e^{-|\tau|}$
$R_X(f) = \int_{\mathbb{R}} e^{-|\tau|} e^{-i2\pi f\tau} d\tau = \int_{-\infty}^{0} e^{(1-i2\pi f)\tau} d\tau + \int_{0}^{\infty} e^{-(1+i2\pi f)\tau} d\tau =$
$= \frac{1}{1-i2\pi f} + \frac{1}{1+2\pi f} = \frac{2}{1+(2\pi f)^2}$

(b) $\sum_{k\in\mathbb{Z}} R_X(f + \frac{k}{d}) = \sum_{k\in\mathbb{Z}} \frac{2}{1+(2\pi(f+k))^2} \approx 2\int_{\mathbb{R}} \frac{dx}{1+(2\pi(f+x))^2} = \left\{ \begin{array}{c} u = 2\pi(f+x) \\ dy = 2\pi\, dx \end{array} \right\} =$
$= \frac{2}{2\pi} \int_{\mathbb{R}} \frac{dx}{1+y^2} = \frac{1}{\pi}[\arctan u]_{-\infty}^{\infty} = \frac{1}{\pi}(\frac{\pi}{2} - (-\frac{\pi}{2})) = 1.$
Thus $R_Z(f) = \left\{ \begin{array}{ll} 1 & -\frac{1}{2} < f \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{array} \right.$

(c) $E(Y_t) = E(X_t + X_{t-1}) = E(X_t) + E(X_{t-1}) = 1 + 1 = 2$
$r_Y(\tau) = C(X_t + X_{t-1}, X_{t+\tau} + X_{t+\tau-1}) = r_x(\tau) + r_X(\tau - 1) + r_X(\tau) + r_X(\tau + 1) =$
$= 2e^{-|\tau|} + e^{-|\tau-1|} + e^{-|\tau+1|}$. When $\tau \geq 0$ we have that
$r(-\tau) = 2e^{-|-\tau|} + e^{-|-\tau-1|} + e^{-|-\tau+1|} = e^{-|\tau|} + e^{-|\tau+1|} + e^{-|\tau-1|} = r(\tau).$
Thus $\{Y_t : t \in \mathbb{Z}\}$ is weakly stationary.

(d) $r_X(\tau) = e^{-|\tau|} \overset{\text{Thm 21}}{\Rightarrow} nV(m_n^*) \approx \sum_{\tau=-\infty}^{\infty} e^{-|\tau|} = 1 + 2\sum_{\tau=1}^{\infty} e^{-\tau} = 1 + 2 \cdot \frac{e^{-1}}{1-e^{-1}} =$
$= \frac{e+1}{e-1} \Rightarrow V(m_8^*) \approx \frac{1}{8} \cdot \frac{e+1}{e-1} = 0.2705$

(e) $m_8 = \frac{1}{8}(1.73 + 1.85 + 2.25 + 1.89 + 0.82 + 1.71 + 2.11) = 1.545.$
Since $\{X_t\}$ is weakly stationary, the expectation function constantly attains one value (in this case estimated to be 1.545) for all $t \in \mathbb{R}$. □

5. Let $\{X_t : t \in \mathbb{R}\}$ be shot noise with intensity $\lambda = 100$ and impulse function

$$g(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the covariance function of $\{X_t\}$. (3p)

(b) Determine the probability $P(X_t + X_{t+0.5} > 100)$. Motivate any approximations.(3p)

Solution:

(a) For $\tau \geq 0$ we have that $r_X(\tau) = \{\text{Campbell's formulae}\} = \int \lambda g(u)g(u-\tau)\,du$ where the integrand is non-zero only when $u - \tau \geq 0$ and $u \leq 1$. Thus $r_X(\tau) =$
$= 100 \int_\tau^1 e^u e^{u-\tau} du$ for $0 \leq \tau \leq 1$. This simplifies to $100e^{-\tau} \int_\tau^1 e^{2u} du =$
$= 100e^{-\tau}(\frac{1}{2}e^2 - \frac{1}{2}e^{2\tau}) = 169.45e^{-\tau} - 50e^{\tau}$. Since $r_X(\tau) = r_X(-\tau)$ we have that

$$r_X(\tau) = \begin{cases} 369.45e^{-|\tau|} - 50e^{|\tau|} & |\tau| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(X_t + X_{t+0.5} > 100) = 1 - P(X_t + X_{t+0.5} \leq 100)$. Since $X_t = \sum_k g(t + \tau_k)$ and $\lambda = 100$, there are $E(X_t) = 100 \int_0^1 e^{-u} du = 100(-e^{-1} + e^0) = 63.2$ emissions per time unit, we may use the central limit theorem for approximating probabilities about X_t with good accuracy:
$X_t + X_{t+0.5}$ is approximately $N(\mu, \sigma^2)$ where
$\mu = E(X_t + X_{t+0.5}) = 2 \cdot 63.2 = 126.4$
$\sigma^2 = V(X_t + X_{t+0.5}) = V(X_t) + V(X_{t+0.5}) + 2C(X_t, X_{t+0.5}) = 2r_X(0) + 2r_X(0.5) =$
$= 2 \cdot 319.45 + 2 \cdot 141.65 = 922.1934$. Thus $P(X_t + X_{t+0.5} > 100) \approx 1 - \Phi(\frac{100-126.4}{\sqrt{922.1934}}) =$
$= \Phi(0.8693) = 0.8078$. $\square$