Solutions to Exam for Random Processes, 5 points/7.5 ECTS

April 16, 2004, 9 am – 1 pm

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe.

1. Show that if a weakly stationary process $\{X_t\}$ is differentiable in squared mean, then the covariance function of the derivative process $\{X_t'\}$ satisfies $r_{X'}(\tau) = -r_X''(\tau)$ for all τ. (3p)

   Lösning: (See the papers, p. 98.) □

2. Show that if $\{X_t : t \in \mathbb{R}\}$ is weakly stationary and $Z_t = X_t$ for all $t = 0, \pm 1, \pm 2, \dots$, then the spectral density function of the process $\{Z_t\}$ is $R_Z(f) = \sum_{k\in\mathbb{Z}} R_X(f + \frac{k}{d})$ where $-\frac{1}{2d} < f \leq \frac{1}{2d}$. (4p)

   Lösning: (See the papers, p. 54.) □

3. Let $\{X_t\}$ be a random process with covariance function $r(\tau) = \sqrt{2\pi}e^{-2\pi^2\tau^2}$. How large error due to the alias effect is risked by sampling $\{X_t\}$ at time-points $t = 0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \dots$? (Hint: Use the table of standard normal percentiles!) (4p)

   Lösning: $r(\tau) = \sqrt{2\pi}e^{-2\pi^2\tau^2} \Rightarrow R(f) = e^{-f^2/2}$.
   "Error due to alias effect" $= \dfrac{4 \text{ "tail-parts"}}{\text{Total effect}} = \dfrac{4\int_{1/2d}^{\infty} R(f)\,df}{\int_{\mathbb{R}} R(f)\,df}$
   $\int_{\mathbb{R}} R(f)\,df = \int_{-\infty}^{\infty} e^{-f^2/2}df$. Now, if $X \in N(0,1)$, then $P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$ the values of which are available by the table of standard normal percentiles. In particular $1 = P(X \leq \infty) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}dt \Rightarrow \int_{-\infty}^{\infty} e^{-f^2/2}df = \sqrt{2\pi}$. Further we have that $\int_{1/2d}^{\infty} R(f)\,df = \int_{-\infty}^{\infty} e^{-f^2/2}df - \int_{-\infty}^{1/2d} e^{-f^2/2}df = \sqrt{2\pi}(1 - \int_{-\infty}^{1/2d} \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt) =$ $= \sqrt{2\pi}(1 - \Phi(\frac{1}{2d}))$ where Φ is the standard normal distribution function. Since $d = \frac{1}{3}$, we have finally that "Error due to alias effect" $= \frac{4\sqrt{2\pi}(1-\Phi(\frac{3}{2}))}{\sqrt{2\pi}} = 4(1-0.9332) = 0.2672$. □

4. Let $\{X_t : t \in \mathbb{R}\}$ be a process with covariance function $r_X(\tau) = 1/(4+\tau^2)$.

   (a) Derive the spectral density function of the derivative process $\{X_t'\}$. (3p)

   (b) Let $\{Y_t\}$ be the process defined by $Y_t = X_t - X_{t-1}$ for all t. Determine the cross-covariance function of $\{X_t\}$ and $\{Y_t\}$: $r_{X,Y}(t, t+\tau) = C(X_t, Y_{t+\tau})$. (3p)

Lösning:

(a) $r_X(\tau) = \frac{1}{4+\tau^2} \;\Rightarrow\; R_X(f) = \frac{\pi}{2}e^{-4\pi|f|} \;\Rightarrow$
$\Rightarrow\; R_{X'}(f) = (2\pi f)^2 R_X(f) = (2\pi f)^2 \frac{\pi}{2} e^{-4\pi|f|} = 2\pi^3 f^2 e^{-4\pi|f|}$

(b) $C(X_t, Y_{t+\tau}) = C(X_t, X_{t+\tau} - X_{t+\tau-1}) = C(X_t, X_{t+\tau}) - C(X_t, X_{t+\tau-1}) =$

$$= r_X(\tau) - r_X(\tau-1) = \frac{1}{4+\tau^2} - \frac{1}{4+(\tau-1)^2} = \frac{4+(\tau-1)^2-4-\tau^2}{(4+\tau^2)(4+(\tau-1)^2)} =$$

$$= \frac{4+\tau^2-2\tau+1-4-\tau^2}{(4+\tau^2)(4+(\tau-1)^2)} = \frac{1-2\tau}{(4+\tau^2)(4+(\tau-1)^2)}$$

□

5. Bob is sometimes at his office and sometimes somewhere else during his working hours. Let $\{X_t\}$ be a Poisson process with intensity 1. Then a process describing wether Bob is in or outside his office at time t is $\{Y_t\}$ where $Y_t = (-1)^{X_t}$. Determine the covariance function of $\{Y_t\}$. (4p)

Lösning: $C(Y_s, Y_t) = E(Y_s Y_t) - E(Y_s)E(Y_t)$.
Since Y_t can only take the values -1 and 1 we have that $E(Y_t) =$
$= (-1)P(Y_t = -1) + 1 \cdot P(Y_t = 1) = (-1)(1 - P(Y_t = 1)) + P(Y_t = 1) = 2P(Y_t = 1) - 1.$
$P(Y_t = 1) = P(X_t \text{ is even}) = \sum_{k=0}^{\infty} P(X_t = 2k) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} e^{-t} = e^{-t} \sum_{k=0}^{\infty} \frac{1}{2} \cdot \frac{t^k + (-t)^k}{k!} =$
$= e^{-t}\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} + \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\right) = e^{-t}\frac{1}{2}(e^t + e^{-t}) = \frac{1}{2}(1 + e^{-2t}).$
Thus $E(Y_t) = 2 \cdot \frac{1}{2}(1 + e^{-2t}) - 1 = e^{-2t}$ and $E(Y_s)E(Y_t) = e^{-2(s+t)}$. Further, $E(Y_s Y_t) =$
$= (-1)P(Y_s Y_t = -1) + 1 \cdot P(Y_s Y_t = 1) = (-1)(1 - P(Y_s Y_t = 1)) + P(Y_s Y_t = 1) =$
$= 2P(Y_s Y_t = 1) - 1$. Now, $P(Y_s Y_t = 1) = P((-1)^{X_s + X_t} = 1) + P(X_s + X_t \text{ even}) =$
$= \sum_{k=0}^{\infty} \frac{(s+t)^{2k}}{(2k)!} e^{-(s+t)} = \frac{1}{2}(1 + e^{-2(s+t)})$ (the last step according to previous calculations).
Thus, $r_Y(s,t) = \frac{1}{2}(1 + e^{-2(s+t)}) - e^{-2(s+t)} = E(Y_s Y_t) - E(Y_s)E(Y_t) = \frac{1}{2}(1 - e^{-2(s+t)})$. □

6. Let $\{X_t\}$ be an $MA(2)$ process satisfying $X_t = a\epsilon_t - \epsilon_{t-2}$ for all $t \in \mathbb{Z}$ where $V(X_t) = 1$ and $\{\epsilon_t\}$ is normally distributed white noise with $V(\epsilon_t) = \sigma_\epsilon^2$.

   (a) What is the variance σ_ϵ^2? (3p)

   (b) What is the probability $P(X_t + X_{t+2} > 1)$ if $a = 3$? (3p)

   (c) Determine $r_X(\tau)$, the covariance function of $\{X_t\}$. (3p)

Lösning:

(a) $1 = V(X_t) = V(a\epsilon_t - \epsilon_{t-2}) = a^2 V(\epsilon_t) + V(\epsilon_{t-2}) = \sigma_\epsilon^2(a^2+1) \Rightarrow \sigma_\epsilon^2 = \frac{1}{1+a^2}$

(b) $a = 3 \Rightarrow P(X_t + X_{t+2} > 1) = P(3\epsilon_t - \epsilon_{t-2} + 3\epsilon_{t+2} - \epsilon_t > 1) = P(3\epsilon_{t+2} + 2\epsilon_t - \epsilon_{t-2} > 1)$
$\{\epsilon_t\}$ independent and $N(0, \frac{1}{3^2+1}) = N(0, 0.1) \Rightarrow$
$E(3\epsilon_{t+2} + 2\epsilon_t - \epsilon_{t-2}) = 0$ and $V(3\epsilon_{t+2} + 2\epsilon_t - \epsilon_{t-2}) = 9V(\epsilon_{t+2}) + 4V(\epsilon_t) + V(\epsilon_{t-2}) = 1.4$
so $P(X_t + X_{t+2} > 1) = 1 - P(3\epsilon_{t+2} + 2\epsilon_t - \epsilon_{t-2} \leq 1) = 1 - \Phi(\frac{1-0}{\sqrt{1.4}}) = 1 - 0.801 = 0.199$

(c)
$$
\begin{aligned}
r(\tau) &= \begin{cases} \sigma_\epsilon^2 \sum_{j-k=\tau} b_j b_k & |\tau| \leq q \\ 0 & |\tau| > q \end{cases} \\
&= \begin{cases} \frac{1}{1+a^2} \sum_{j=k} b_j^2 & \tau = 0 \\ \frac{1}{1+a^2} \sum_{|j-k|=1} b_j b_k & \tau = \pm 1 \\ \frac{1}{1+a^2} \sum_{|j-k|=2} b_j b_k & \tau = \pm 2 \\ 0 & |\tau| \geq 3 \end{cases} \\
&= \begin{cases} \frac{1}{1+a^2}(1^2 + a^2) & \tau = 0 \\ \frac{1}{1+a^2}(1 \cdot 0 + 0 \cdot a) & \tau = \pm 1 \\ \frac{1}{1+a^2}(1 \cdot a) & \tau = \pm 2 \\ 0 & |\tau| \geq 3 \end{cases} \\
&= \begin{cases} 1 & \tau = 0 \\ \frac{a}{1+a^2} & \tau = \pm 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

□