Lösningar till tentamen i Stokastiska Processer, 5 poäng

January 16, 2004, 13.30 – 17.30

Max number of points: 30. **Bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that a weakly stationary Gaussian process is strongly stationary. (3p)

   Solution: Consider the Gaussian process $\{X_t\}$ at the n time-points $t_1, \ldots, t_n$. If each t_k is increased a constant amount τ, then it follows by weak stationarity that the expectations and covariances are not changed. But the n-dimensional normal distribution is uniquely determined by the expectations and covariances. Thus not only these entities but the entire distribution of $X_{t_1+\tau}, \ldots, X_{t_n+\tau}$ remains unchanged, i.e. the same as that of $X_{t_1}, \ldots, X_{t_n}$. This is to say that the process is strongly stationary. □

2. Let the process $\{X_t\}$ be defined by $X_t = bX_{t-1} + \epsilon_t$ for all $t \in \mathbb{Z}$ where $\{\epsilon_t\}$ is an independent sequence where $\epsilon_t \in N(0, \sigma_\epsilon^2)$. Show that if $\{X_t\}$ is weakly stationary, then $|b| \leq 1$. (3p)

   Solution: Taking the variance of both sides of the defining equality we have that

   $$V(X_t) = V(bX_{t-1} + \epsilon_t) = b^2 V(X_{t-1}) + \sigma_\epsilon^2$$

   since $\{\epsilon_t\}$ being a sequence of independent variables means that $\epsilon_t \perp X_s$ for all $s \leq t-1$. Now, since $\{X_t\}$ is weakly stationary, this means that

   $$r_X(0) = b^2 r_X(0) + \sigma_\epsilon^2$$

   i.e. $r_X(0)(1-b^2) = \sigma_\epsilon^2$. But both $r_X(0) \geq 0$ and $\sigma_\epsilon^2 \geq 0$, so $b^2 \leq 1$ or, equivalently, $|b| \leq 1$. □

3. Let $\{X_t : t \in \mathbb{R}\}$ be shot noise with intensity 35 and impulse function

   $$g(t) = \begin{cases} 2t^3 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

   Calculate

   (a) expectation and covariance function of $\{X_t\}$. (3p)

   (b) approximately the probability $P(X_{100} > 20)$. (Motivate any approximations!) (3p)

Solution:

(a) $m = \lambda \int_{\mathbb{R}} g(u)\,du = 35 \cdot 2[\frac{u^4}{4}]_0^1 = 17.5$
$r(\tau) = \lambda \int_{\mathbb{R}} g(u)g(u-\tau)\,du$
This integrand is zero whenever either $u \notin [0,1]$ or $u-\tau \notin [0,1]$, i.e. it is non-zero whenever both $0 \le u \le 1$ and $0 \le u-\tau \le 1$, i.e. when $0 \le u \le 1$ and $\tau \le u \le 1+\tau$, i.e. when $\tau \le u \le 1$ (assuming $\tau > 0$ since $r(\tau)$ is symmetrical). Thus we have that

$$\begin{aligned} r(\tau) &= 35 \int_\tau^1 2u^3 \cdot 2(u-\tau)^3 du \\ &= 35 \cdot 4 \int_\tau^1 u^3(u^3 - 3u^2\tau + 3u\tau^2 - \tau^3)\,du \\ &= 140[\tfrac{u^7}{7} - 3\tfrac{u^6}{6}\tau + 3\tfrac{u^5}{5}\tau^2 - \tfrac{u^4}{4}\tau^3]_\tau^1 \\ &= \tau^7 - 35\tau^3 + 84\tau^2 - 70\tau + 20 \end{aligned}$$

(and for $\tau \le 0$ we have that $r(\tau) = r(-\tau)$.)

(b) Since $X_t = \sum_k g(t-\tau_k)$ and there are in the mean $\lambda = 35$ pulses per time unit, X_{100} is the sum of many pulses, and thus X_{100} is approximately $N(\mu, \sigma^2)$ where $\mu = m = 17.5$ and $\sigma^2 = r(0) = 20$ and therefore $P(X_{100} > 20) \approx 1 - \Phi(\frac{20-17.5}{\sqrt{20}}) =$
$= 1 - \Phi(0.56) = 0.2877$.

□

4. For which values of α is $r(\tau)$ a covariance function in discrete time if

$$r(\tau) = \begin{cases} \alpha & \text{if } \tau = 0 \\ \alpha^2 & \text{if } |\tau| = 3 \\ 0 & \text{o.w.} \end{cases}$$

(3p)

Solution: We will calculate the spectral density function and see for which values of α the three conditions symmetry, integrability, positivity are satisfied. We have that $R(f) = r(0) + r(1)(e^{-i2\pi f\cdot(-3)} + e^{-i2\pi f \cdot 3} = \alpha + 2\alpha^2\cos(6\pi f)$, $-\frac{1}{2} \le f < \frac{1}{2}$. This function is symmetric and integrable for all $\alpha \in \mathbb{R}$. But what about the positivity?

$$\alpha + 2\alpha^2\cos(6\pi f) > 0 \quad \Leftrightarrow \quad \begin{cases} \alpha - 2\alpha^2 > 0 \\ \alpha + 2\alpha^2 > 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha(1-2\alpha) > 0 \\ \alpha(1+2\alpha) > 0 \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \{\alpha > 0 \text{ and } 1-2\alpha > 0\} \text{ or } \{\alpha < 0 \text{ and } 1-2\alpha < 0\} \\ \{\alpha > 0 \text{ and } 1+2\alpha > 0\} \text{ or } \{\alpha < 0 \text{ and } 1+2\alpha < 0\} \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \{\alpha > 0 \text{ and } \alpha < \frac{1}{2}\} \text{ or } \{\alpha < 0 \text{ and } \alpha > \frac{1}{2}\} \\ \{\alpha > 0 \text{ and } \alpha > -\frac{1}{2}\} \text{ or } \{\alpha < 0 \text{ and } \alpha < -\frac{1}{2}\} \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} (1): \ 0 < \alpha < \frac{1}{2} \\ (2): \ \alpha > 0 \text{ or } \alpha < -\frac{1}{2} \end{cases}$$

(1) and (2) $\Leftrightarrow \ 0 < \alpha < \frac{1}{2}$.

□

5. Let $\{X_t : t \in \mathbb{R}, t \ge 0\}$ be a Poisson process with intensity 1. Construct the process $\{Y_t\}$ by $Y_t = 0.5^t X_{4^t} - 2^t$ for all $t \in \mathbb{R}$.

(a) Show that $\{Y_t\}$ is weakly stationary. (3p)

(b) Calculate $P(2Y_1 = Y_0 + 1)$. (3p)

Solution:

(a) Remember that for a Poisson process $\{X_t\}$ with intensity λ we have that $E(X_t) = \lambda t$ and $C(X_s, X_t) = \lambda \min(s,t)$. Thus
$m_Y = E(0.5^t X_{4^t} - 2^t) = 0.5^t E(X_{4^t}) - 2^t = 0.5^t \cdot 1 \cdot 4^t - 2^t = 0$
$r_Y(s,t) = C(Y_s, Y_t) = C(0.5^s X_{4^s} - 2^s\ ,\ 0.5^t X_{4^t} - 2^t) = 0.5^{s+t} C(X_{4^s}, X_{4^t}) =$
$= (\frac{1}{2})^{s+t} \min(4^s, 4^t) = \min((\frac{1}{2})^{s+t} 4^s, (\frac{1}{2})^{s+t} 4^t) = \min(2^{-(s+t)} 2^{2s}, 2^{-(s+t)} 2^{2t}) =$
$= \min(2^{s-t}, 2^{t-s}) = 2^{\min(s-t, t-s)} = 2^{-\max(s-t,t-s)} = 2^{-|s-t|}$
Since the ef is 0 and the cvf is a function only of $s-t$, the process $\{Y_t\}$ is weakly stationary.

(b)
$$\begin{aligned} P(2Y_1 = Y_0 + 1) &= P(2(0.5^1 X_{4^1} - 2^1) - (0.5^0 X_{4^0} - 2^0) = 1) \\ &= P(2 \cdot 0.5 X_4 - X_1 = 1 + 2 \cdot 2 - 1) \\ &\quad \{\, X_4 - X_1 \in Poi(4-1) \,\} \\ &= \frac{3^4}{4!} e^{-3} \\ &= 0.168 \end{aligned}$$

□

6. Let $\{X_t\}$ be a weakly stationary process with $r_X(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau}$ and construct $\{Y_t\}$ by $Y_t = \theta X_t + (1-\theta) X_{t-1}$ for all $t \in \mathbb{R}$ where $\theta \in (0,1)$.

(a) Determine the impulse response which filters $\{X_t\}$ into $\{Y_t\}$. (3p)

(b) Derive the spectral density of the derivative process $\{Y_t'\}$. (3p)

(c) Consider only the frequency $f = \frac{1}{2}$. For which value of θ is the spectral density of Y_t minimized (at $f = \frac{1}{2}$)? (3p)

Solution:

(a) We shall find a function $h(u)$ such that $\int_{\mathbb{R}} h(t-u) X_u \, du = \theta X_t + (1-\theta) X_{t-1}$. For getting the X_t term we try with $\delta_t(u)$, and for the X_{t-1} term, try $\delta_{t-1}(u)$:
Let $h(t-u) = \theta \delta_t(u) + (1-\theta)\delta_{t-1}(u)$. Then $h(v) = \theta\delta(v) + (1-\theta)\delta(v-1)$ and checking this by integration yields $\int_{\mathbb{R}} h(t-u) X_u \, du =$
$= \theta \int_{\mathbb{R}} \delta(t-u) X_u \, du + (1-\theta) \int_{\mathbb{R}} \delta(t-u-1) X_u \, du = \theta X_t + (1-\theta) X_{t-1}$ (ok!)

(b) $H(f) = \int_{\mathbb{R}} (\theta\delta(t) + (1-\theta)\delta(t-1)) e^{-i2\pi f t} dt = \theta e^0 + (1-\theta) e^{-i2\pi f} =$
$= \theta + (1-\theta)\cos(2\pi f) - i(1-\theta)\sin(2\pi f)$
and $|H|^2 = (\theta + (1-\theta)\cos(2\pi f))^2 + (-(1-\theta)\sin(2\pi f))^2 = \theta^2 + 2\theta(1-\theta)\cos(2\pi f) + 1$.
Further, $R_X(f) = \begin{cases} 1/2 & \text{if } |f| \le 1 \\ 0 & \text{o.w.} \end{cases}$ (according to the table) which means that
$$R_Y(f) = |H(f)|^2 R_X(f) = \begin{cases} \frac{1}{2}(\theta^2 + 1) + \theta(1-\theta)\cos(2\pi f) & \text{if } |f| \le 1 \\ 0 & \text{o.w.} \end{cases}$$
i.e. $$R_{Y'}(f) = (2\pi f)^2 R_Y(f) = \begin{cases} 2\pi^2 f^2 (\theta^2 + 1 + 2\theta(1-\theta)\cos(2\pi f)) & \text{if } |f| \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(c) $\frac{d}{d\theta} R_Y(\frac{1}{2}) = \theta + (1-2\theta)\cos(2\pi \frac{1}{2}) = 3\theta - 1$. Solving $\frac{d}{d\theta} R_Y(\frac{1}{2}) = 0$ with respect to θ gives $\theta = \frac{1}{3}$. The second derivative $\frac{d^2}{d\theta^2} R_Y(\frac{1}{2}) = 3 > 0$ which means that $R_Y(\frac{1}{2})$ is strictly convex in θ meaning that $\theta = \frac{1}{3}$ is indeed a minimum of $R_Y(\frac{1}{2})$.

□