Solutions to Exam for Random Processes, 5 points/7.5 ECTS

January 4, 2005, 9 am – 1 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Assume that $\{X_t\}$ is a weakly stationary process with cvf r_X, and that $r_X = G$ where G is the Fourier transform of g. Show that the spectral density function of $\{X_t\}$ is $R_X = g$. (3p)

   Solution: Since $G = \mathcal{F}(g)$ we have that

   $$g(-\tau) = \mathcal{F}^{-1}(G)(-\tau) = \int_{\mathbb{R}} e^{i2\pi f\cdot(-\tau)}G(f)\,df = \int_{\mathbb{R}} e^{-i2\pi f\tau}G(f)\,df = \mathcal{F}(G)(\tau)$$

   Now, if $r_X = G$, then

   $$R_X(f) = \mathcal{F}(r_X)(f) = \mathcal{F}(G)(f) = g(-f)$$

   But because r_X is cvf of a weakly stationary process, it is even (i.e. $r_X(\tau) = r_X(-\tau)$ for all τ) which means that also the spectral density is an even function. Thus $R_X(f) = g(f)$ for all f. □

2. Show that a weakly stationary Gaussian process is strongly stationary. (3p)

   Solution: Weak stationarity ⇒ $E(X_t) = m$ and $C(X_t, X_{t+h}) = r(h)$. Now, given any timepoints $t_1, t_2, \ldots, t_n$ and time difference h consider the vectors $\boldsymbol{X}_t = [X_{t_1}, X_{t_2}, \ldots, X_{t_n}]$ and $\boldsymbol{X}_{t+h} = [X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h}]$. To prove strong stationarity we must show that $\boldsymbol{X}_t \stackrel{\mathcal{D}}{=} \boldsymbol{X}_{t+h}$. Since the process is weakly stationary Gaussian process, each subset of process variables are multivariate normally distributed with expectation vector $\boldsymbol{\mu}$ and covariance matrix Σ. The expectation vector $\boldsymbol{\mu}$ is

   $$E(\boldsymbol{X}_t) = E\Big([X_{t_1}, X_{t_2}, \ldots, X_{t_n}]\Big) = [m, m, \ldots, m] = E\Big([X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h}]\Big) = E(\boldsymbol{X}_{t+h})$$

   i.e. the expectation vector is the same for $\boldsymbol{X}_{t+h}$. The covariance matrix Σ_t of $\boldsymbol{X}_t$ is a matrix with elements $r_{ij} = C(X_{t_i}, X_{t_j})$ where $i, j = 1, 2, \ldots, n$. Since the process is weakly stationary $r_{ij} = r(t_i - t_j)$. In the same way the covariance matrix Σ_{t+h} of $\boldsymbol{X}_{t+h}$ is a matrix with elements $r_{ij,h} = C(X_{t_i+h}, X_{t_j+h})$ where $i, j = 1, 2, \ldots, n$. Since the process is weakly stationary $r_{ij,h} = r(t_i+h-(t_j+h)) = r(t_i - t_j) = r_{ij}$ which means that that $\Sigma_t = \Sigma_{t+h}$. But since a mutivariate normally distributed random variable is completely determined by its expectation and covariance, the distributions of $\boldsymbol{X}_t$ and of $\boldsymbol{X}_{t+h}$ are the same. □

3. Suppose $\{X_t\}$ is a weakly stationary process with ef $m_X = 1$ and spectral density function $R_X(f) = \sqrt{\frac{1}{4}e^{-|f|}}$.

(a) Determine the cvf $r_X(\tau)$. (2p)

(b) Assuming that $\{X_t\}$ is a Gaussian process, calculate the probability $P(X_t + X_t' \le 5)$. (4p)

Solution:

(a)
$$R_X = \sqrt{\frac{1}{4}\,e^{-|f|}} = \frac{1}{2}\,e^{-\frac{1}{2}|f|} \quad \Rightarrow \quad r_X = \frac{1}{2}\cdot\frac{2\cdot\frac{1}{2}}{(\frac{1}{2})^2 + (2\pi\tau)^2} \;=\; \frac{2}{1+(4\pi\tau)^2}$$

(b) $\{X_t\}$ Gaussian $\Rightarrow$ $\{X_t'\}$ Gaussian $\Rightarrow$ $\{X_t + X_t'\}$ Gaussian

$$\begin{aligned}
r_{X'} &= -r_X'' \\
&= -\tfrac{d^2}{d\tau^2}\left(\tfrac{2}{1+(4\pi)^2\tau^2}\right) \\
&= -\tfrac{d}{d\tau}(2\cdot(-1)(1+(4\pi)^2\tau^2)^{-2}(4\pi)^2\cdot 2\tau) \\
&= -\tfrac{d}{d\tau}\left(-\tfrac{(8\pi)^2\tau}{(1+(4\pi)^2\tau^2)^2}\right) \\
&= \tfrac{(8\pi)^2(1+(4\pi)^2\tau^2)^2-(8\pi)^2\tau 2(1+(4\pi)^2\tau^2)(4\pi)^2\cdot 2\tau}{(1+(4\pi)^2\tau^2)^4} \\
&= \left(\tfrac{8\pi}{1+(4\pi\tau)^2}\right)^2\left(1-\tfrac{(8\pi\tau)^2}{1+(4\pi\tau)^2}\right)
\end{aligned}$$

$E(X_t + X_t') = E(X_t) + E(X_t') = 1 + 0 = 1.$
$C(X_t, X_t') = 0 \Rightarrow V(X_t + X_t') = r_X(0) + r_{X'}(0) = 2 + (8\pi)^2.$
$P(X_t + X_t' > 5) = 1 - \Phi\left(\frac{5-1}{\sqrt{2-(8\pi)^2}}\right) = 1 - \Phi(0.159) = 0.4364.$ □

4. Let $r_X(\tau) = \sum_{k\in\mathbb{Z}} \dfrac{\delta_k(\tau)}{2^{|\tau|}}$ be the cvf of the weakly stationary process $\{X_t : t \in \mathbb{R}\}$.

(a) Calculate $R_X(0)$. (4p)

(b) Suppose that $\{X_t\}$ is filtered with the response function $h(t) = \delta_c(t)$ (where $c \in \mathbb{R}$) resulting in the output process $\{Y_t\}$. Calculate the cvf r_Y. (4p)

Solution:

(a) $r_X(\tau) = \sum_{k\in\mathbb{Z}} \frac{\delta_k(\tau)}{2^{|\tau|}}$. $\quad R_X(f) = \int_{\mathbb{R}} r_X(\tau) e^{-i2\pi f\tau} d\tau \Rightarrow$
$\Rightarrow \quad R_X(0) = \int_{\mathbb{R}} r_X(\tau)\,d\tau = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{\delta_k(\tau)}{2^{|\tau|}}\,d\tau = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} 2^{-|\tau|}\delta_k(\tau)\,d\tau =$
$= \sum_{k\in\mathbb{Z}} 2^{-|k|} = \sum_{k=-\infty}^{-1} 2^k + 2^0 + \sum_{k=1}^{\infty} 2^{-k} = 1 + 2\sum_{k=1}^{\infty} 2^{-k} =$
$= 1 + \sum_{k=0}^{\infty} 2^{-k} = 1 + \frac{1}{1-\frac{1}{2}} = 3.$

(b) $r_Y(\tau) = \iint_{\mathbb{R}^2} h(u)h(v)r_X(\tau+u-v)\,du\,dv = \iint_{\mathbb{R}^2} \delta_c(u)\delta_c(v) \sum_{k\in\mathbb{Z}} \frac{\delta_k(\tau+u-v)}{2^{|\tau+u-v|}}\,du\,dv =$
$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \delta_c(u)\delta_c(v) \frac{\delta_k(\tau+u-v)}{2^{|\tau+u-v|}}\,du\right) dv = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \delta_c(v) \frac{\delta_k(\tau+c-v)}{2^{|\tau+c-v|}}\,dv =$
$= \sum_{k\in\mathbb{Z}} \frac{\delta_k(\tau+c-c)}{2^{|\tau+c-c|}} = \sum_{k\in\mathbb{Z}} \frac{\delta_k(\tau)}{2^{|\tau|}} = r_X(\tau)$ □

5. Let $\{X_t\}$ be an $MA(q)$ process defined by $X_t = \sum_{k=0}^{q}(-1)^k \epsilon_{t-k}$ where $\sigma_\epsilon^2 = 1$.

(a) Calculate the cvf r_X. (4p)

(b) What kind of process $\{Y_t\}$ is achieved by by letting $Y_t = X_t + X_{t-1}$. (3p)

(c) Calculate the spectral density function R_Y in the case when $q = 99$. (3p)

Solution:

(a) $X_t = \epsilon_t - \epsilon_{t-1} + \epsilon_{t-2} - \ldots + (-1)^q \epsilon_{t-q}$.

$r_X(\tau) = \begin{cases} \sigma_\epsilon^2 \sum_{j-k=\tau} b_j b_k & |\tau| \leq q \\ 0 & |\tau| > q \end{cases}$

Since $b_j = (-1)^j$ and $\sigma_\epsilon^2 = 1$ we have that

$r_X(0) = \sum_{j=0}^{q} (-1)^{2j} = q + 1$

$r_X(1) = \sum_{j=0}^{q} (-1)^{j+j-1} = -q$

$\vdots$

$r_X(q) = (-1)^q$

$\Rightarrow \; r_X(\tau) = \begin{cases} (-1)^\tau (q + 1 - |\tau|) & |\tau| \leq q \\ 0 & \text{o.w.} \end{cases}$

(b) $Y_t = X_t + X_{t-1} =$

$= \epsilon_t - \epsilon_{t-1} + \epsilon_{t-2} - \ldots + (-1)^q \epsilon_{t-q} + \epsilon_{t-1} - \epsilon_{t-2} + \ldots + (-1)^{q-1} \epsilon_{t-q} + (-1)^q \epsilon_{t-q-1}$

$= \epsilon_t + (-1)^q \epsilon_{t-q-1}$

which apparently is an $MA(q+1)$ process with parameters

$c_0 = 1, c_1 = 0, \ldots, c_q = 0, c_{q+1} = (-1)^q$.

(c)

$$\begin{aligned} R_Y(f) &= \sigma_\epsilon^2 |\sum_{k=0}^{q+1} c_k e^{-i2\pi f k}|^2 \\ &\overset{\{q=99\}}{=} |e^0 + (-1)^{99} e^{-i2\pi f(99+1)}|^2 \\ &= (1 - \cos(2\pi f 100))^2 + (-\sin(2\pi f 100))^2 \\ &= 1 - 2\cos(200\pi f) + \cos^2(200\pi f) + \sin^2(200\pi f) \\ &= 2(1 - \cos(200\pi f)) \end{aligned}$$

□