# Solutions to Exam for Random Processes, 5 points/7.5 ECTS

March 15, 2005, 1.30 pm – 5.30 pm

**Max number of points**: 30.
**Halmstad University grading bounds**: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
**ECTS bounds**: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
**Allowed aids**: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
**Examiner**: Eric Järpe (035-16 76 53).

1. Show that if $\{X_t : t \in \mathbb{R}\}$ is weakly stationary, $Z_t = X_t$ for $t = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$, then
$$R_Z(f) = \sum_{k\in\mathbb{Z}} R_X(f + 2k) \quad \text{where } -1 < f \le 1 \qquad (4p)$$

   **Solution:** (See the compendium, page 56.) □

2. Show the Yule-Walker equations, i.e. if $\{X_t\}$ is an $AR(p)$ process, then
$$\sum_{k=0}^{p} a_k r_X(\tau - k) = \begin{cases} \sigma_\epsilon^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau = 1, 2, \ldots \end{cases} \qquad (3p)$$

   **Solution:** (See the compendium, page 88–89.) □

3. A car drives on a trip for several hours. During hour $k$ it drives with speed $X_k$ km/h independently of speed during previous hours, where $k = 1, 2, 3, \ldots$, and $X_k \in Poi(50)$. Let $Y_t$ be the distance driven by time $t$ where $t$ is measured in hours from the start. Calculate

   (a) $V(X_4)$. (2p)

   (b) $C(Y_3, Y_4)$. (4p)

   (c) approximately $P(200 \le Y_4 \le 220)$. (2p)

   **Solution:**

   (a) $X_4 \in Poi(50) \;\Rightarrow\; V(X_4) = 50$.

   (b) Since $Y_t$ is the sum of independent equally Poisson distributed variables, the process $\{Y_t\}$ is a Poisson process. Thus the cvf of $\{Y_t\}$ is $r_Y(s,t) = 50\min(s,t)$, which means that $C(Y_3, Y_4) = r_Y(3,4) = 50\min(3,4) = 150$.

   (c) $Y_4 = \sum_{s=1}^{4} X_s$ where $X_s$ are independent and $Poi(50)$
   $= \sum_{k=1}^{200} Z_k$ where $Z_k$ are independent and $Poi(1)$.
   Thus according to the Central limit theorem, $Y_4 \overset{appr.}{\in} N(4\cdot 50, \sqrt{4\cdot 50}) = N(200, 14.142) \;\Rightarrow\; P(200 \le Y_4 \le 220) = P(Y_4 \le 220) - P(Y_4 \le 220) = \Phi(\frac{220-200}{14.142}) - \Phi(0) = \Phi(1.4142) - 0.5 = 0.4207$. □

4. Assume $\{X_t : t \in \mathbb{R}\}$ is weakly stationary and has cvf $r_X(\tau) = 1/(1+\tau^2)$. Determine the cross-cvf with the derivative process $r_{X,X'}(s,t)$. (3p)

**Solution:** $r_X(\tau) = \dfrac{1}{1+\tau^2}$ $\quad r'_X(\tau) = -\dfrac{2\tau}{(1+\tau^2)^2}$

$r_{X,X'}(s,t) = r'_X(s,t) = -\dfrac{2(s-t)}{(1+(s-t)^2)^2}$. □

5. An $MA(2)$ process $\{X_t : t \in \mathbb{Z}\}$ is defined by $X_t = \epsilon_t + \epsilon_{t-2}$ where $V(\epsilon_t) = 1$. Calculate its spectral density function $R_X(f)$. (4p)

**Solution:**

$$\begin{aligned}
R_X(f) &= 1 \cdot \left| \sum_{k=0}^{2} b_k e^{-i2\pi f k} \right|^2 \\
&= |1 \cdot e^{-i2\pi f \cdot 0} + 0 + 1 \cdot e^{-i2\pi f \cdot 2}|^2 \\
&= |1 + \cos(4\pi f) - i\sin(4\pi f)|^2 \\
&= (1+\cos(4\pi f))^2 + \sin^2(4\pi f) \\
&= 1 + 2\cos(4\pi f) + \cos^2(4\pi f) + \sin^2(4\pi f) \\
&= 2(1+\cos(4\pi f)) \quad \text{for } f \in (-\tfrac{1}{2}, \tfrac{1}{2}]
\end{aligned}$$

□

6. Suppose $\{X_t : t \in \mathbb{R}\}$ with spectral density $e^{-|f|}$ is filtered with impulse response $\delta_{-1}(t) + \delta_1(t)$. What is the spectral density $R_Y(f)$ of the output signal $\{Y_t\}$?(3p)

**Solution:**
Since the impulse response is $h(t) = \delta_{-1}(t) + \delta_1(t)$, the transfer function $H(f)$ is its Fourier transform $\int_{\mathbb{R}} e^{-i2\pi f t}(\delta_{-1}(t) + \delta_1(t))\,dt = e^{i2\pi f} + e^{-i2\pi f} = 2\cos(2\pi f)$. Thus the spectral density of the output process $\{Y_t\}$ is $R_Y(f) = |H|^2 R_X = 4\cos^2(2\pi f)e^{-|f|}$. □

7. Let $\{X_t\}$ be an $MA(q)$ process with $b_k = (-1)^k$ and $\sigma_\epsilon = 1$. Observations of $\{X_t\}$ are going to be made and, based on these observations, the ef $m_X$ will be estimated. Assuming that $q$ is odd, calculate approximately how many observations that are needed to have $V(m_n^*) < \frac{1}{4}$. (5p)

**Solution:**

$$r_X(\tau) = \begin{cases} \sigma_\epsilon^2 \sum_{j-k=\tau} b_j b_k & |\tau| \le q \\ 0 & |\tau| > q \end{cases} = \begin{cases} (-1)^\tau (q+1-|\tau|) & |\tau| \le q \\ 0 & \text{o.w.} \end{cases}$$

To be able to use the theorem about approximation of $V(m_n^*)$, we must first check that the condition $\sum_{\tau\in\mathbb{Z}} |r_X(\tau)| < \infty$ is satisfied. We have that $\sum_{\tau\in\mathbb{Z}} |r_X(\tau)| = \sum_{\tau=-q}^{q} |(-1)^\tau (q+1-|\tau|)| = (2q+1)(q+1) - 2\sum_{\tau=1}^{q} \tau (2q+1)(q+1) - 2\frac{q(q+1)}{2} = (q+1)^2 < \infty$. Now we may use the approximation of $V(m_n^*)$ to deduce that $\frac{1}{4} > V(m_n^*) \approx \frac{1}{n} \lim_{k\to\infty} kV(m_k^*) = \frac{1}{n}\sum_{\tau\in\mathbb{Z}} r_X(\tau) = \frac{1}{n}\sum_{\tau=-q}^{q} (-1)^\tau (q+1-|\tau|) = \frac{1}{n}(1 + 2\sum_{\tau=1}^{q}(-1)^\tau (q+1-|\tau|)) \overset{q \text{ odd}}{=} \frac{1}{n}(1 + 2\sum_{\tau=1}^{q}(-1)^\tau \tau) = \frac{1}{n}(1 + 2(\underbrace{\sum_{\tau=1,3,\ldots,q}(-1)^\tau \tau}_{I} + \underbrace{\sum_{\tau=2,4,\ldots,q-1}(-1)^\tau \tau}_{II}))$

$I = \sum_{\tau=1,3,\ldots,q}(-\tau) = -\sum_{j=0}^{(q-1)/2}(2j+1) = -2\sum_{j=0}^{(q-1)/2} j - \frac{q-1}{2} = = -2\cdot\frac{\frac{q-1}{2}\cdot(\frac{q-1}{2}+1)}{2} - \frac{q-1}{2} = -\frac{q^2-1}{4} - \frac{2q-2}{4} = -\frac{(q-1)^2}{4}$

$II = \sum_{\tau=2,4,\ldots,q-1}\tau = 2\sum_{j=1}^{(q-1)/2} j = 2\cdot\frac{\frac{q-1}{2}\cdot(\frac{q-1}{2}+1)}{2} = \frac{q^2-1}{4}$

$\Rightarrow V(m_n^*) \approx \frac{1}{n}(1 + 2(\frac{(q-1)^2+q^2-1}{4})) = \frac{q}{n} \Rightarrow n > 4q$ (i.e. $n \ge 397$ if $q = 99$). $\square$